Directorate of Distance and Continuing Education

Manonmaniam Sundaranar University

Tirunelveli-627 012, Tamil Nadu.

M.A. ECONOMICS

(First Year)

MATHEMATICAL ECONOMICS

SECM23

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MATHEMATICAL ECONOMICS

Course Objectives:

1. The paper aims to introduce students to the basic building blocks of mathematical analysis used in modern economic theory.
2. To equip the students with mathematical tools and to optimize both static and dynamic economic environment.

UNIT I: INTRODUCTION TO LINEAR ALGEBRA

Sets-Basic concepts-Ordered sets-Relations-Order relations-Metric Spaces-open and closed sets– Convergence - Linear Algebra , Vectors, matrices, inverse, simultaneous linear equations, Cramer’s rule for solving system of linear equations, input-output model, Hawkins - Simon condition, open and closed models , quadratic equation, characteristic (eigen) roots and vectors

UNIT II: DIFFERENTIAL CALCULUS

Introduction to Functions, Limits and Continuity, Derivatives –Concept ofmaxima& minima, elasticity and point of inflection. Profit & revenue maximization under perfect competition, under monopoly.Maximizing excise tax revenue in monopolistic competitive market, Minimization of cost etc.

UNIT III: OPTIMIZATION TECHNIQUES WITH CONSTRAINTS

Functions of several variables, Partial and total, economic applications, implicit function theorem, higher order derivatives and Young’s theorem, properties of linear homogenous functions, Euler's theorem, Cobb – Douglas Production Function - Constrained Optimization-Lagrangian Multiplier Technique- Vector and Matrix Differentiation - Jacobian and Hessian Matrices- Applications-Utility maximization, Profit maximization and Cost minimization.

UNIT IV: LINEAR AND NON-LINEAR PROGRAMMING

Optimization with Inequality Constraints- Linear Programming–Formulation-Primal and Dual- Graphical and Simplex method-Duality Theorem-Non-Linear Programming-Kuhn-Tucker Conditions- Economic Applications.

UNIT V: ECONOMIC DYNAMICS

Differential Equations-Basic Ideas-Types-Solution of Differential Equations (Homogenous and Exact)-Linear Differential Equations with Constant Coefficients (First and Second Order)- Applications- Solow’s Model-Harrod-Domar Model-Applications to Market models- Difference Equations - Types-Linear Difference Equations with Constant Coefficients (First and Second order) and solutions – Applications- Samuelson’s Accelerator-Multiplier model-Cobweb model.

Textbooks:

1. Geoff Renshaw,(2016) Maths for Economics, 4E Oxford University Press.
2. Mabbet A J (1986) Workout Mathematics for Economists, Macmillan Master Series, 4^{th} Edition London.

UNIT I

INTRODUCTION TO LINEAR ALGEBRA

Set Theory

A set is a mathematical model for a collection of objects; it contains elements or members, which can be any mathematical object: numbers, symbols, points in space, lines, other geometrical structures, variables, or even other sets. The empty set is a set with no elements, while a singleton is a set with only one element. If two sets have exactly the same elements, they are equivalent.

Set theory is a fundamental branch of mathematics that deals with the study of sets, which are collections of objects. It is a foundational theory for various other mathematical disciplines and provides a basis for understanding mathematical concepts and structures. Here’s an overview of the key concepts in set theory:

Basic About Set Theory:-

Set theory is an area of mathematics that teaches us about sets and their properties. A set can be defined as a group of objects or a collection of objects. These items are frequently referred to as set elements or members. A set of cricket players, for example, is a group of players.

Because a cricket team can only have 11 players at any given moment, we can claim that this set is finite. A collection of English vowels is another example of a finite set. However, many sets, such as a set of natural numbers, a set of whole numbers, a set of real numbers, a set of imaginary numbers, and so on, have infinite members.

Origin of Set Theory:-

Georg Cantor (1845-1918), a German mathematician, was the first to propose the concept of “Set Theory.” He came across sets while working on “Problems on Trigonometric Series.” It will be impossible to explain other concepts like relations, functions, sequences, probability, geometry, and so on without first comprehending sets.

Definition of Set:-

A set is a well-defined collection of objects or people, as we learnt in the introduction. Many real-life instances of sets include the number of rivers in India, the number of colours in a rainbow, and so on.

Example:-

Consider the following example to better comprehend sets. While walking to school from home, Niva decided to jot down the names of nearby restaurants. The restaurants were listed in the following order:

List 1 : R1 R2 R3 R4 R5

The list above is made up of many objects. It's also well-defined. By well-defined, we mean that anyone should be able to determine whether or not an item belongs to a specific collection. A stationery store, for example, cannot be included in the restaurant category. A set is defined as a well-defined collection of items.

The elements of a set are the objects that make up the set. A set of elements can be finite or infinite. Niva wanted to double-check the list she had created earlier on her way home from school. She wrote the list in the order in which the eateries arrived this time. The revised list was as follows:

List 2 : R5 R4 R3 R2 R1

This is an entirely distinct list. But if we consider the set, then it is the same as the set of list 1. In sets, the order of the elements has no bearing, hence the set remains the same.

Representation of Sets:-

There are two ways to express sets:

1. Roaster form

Example :- A = { a, o, i, u, e }

S = { -1 , 1}

2. Set Builder form

Example :- A = { x : x is a vowel in English alphabet}

S = { x : x^2-1 = 0}

Types of Sets:-

The sets are further divided into categories based on the components or types of elements they contain. Basic set theory distinguishes between the following sorts of sets:

Finite Set: The set with a finite number of elements is called a finite set.

Infinite Set: The set with an infinite number of elements is called an infinite set.

Empty set: The set which has no elements is said to be an empty set.

Singleton set: The set containing only one element is a singleton set.

Equal Set: If two sets have the same elements, they are equal.

Equivalent Set: If cardinality of two sets is equal, they are equivalent sets.

Power Sets: A power set is a collection of all conceivable subsets.

Universal Set: Any set that contains all the sets under discussion is referred to as a universal set.

Subset: A is a subset of B when all of the elements of set A belong to set B.

Set Operations:-

The following are the four most commonly used set operations:

Union of Sets ($\cup$)
Intersection of Sets ($\cap$)
Complement of Sets (‘)
Difference of sets (–)

Set Theory Formulas:-

$n(A \cup B) = n(A) + n(B) - n (A \cap B)$

$n(A\cup B)=n(A)+n(B)$ {when A and B are disjoint sets}

$n(U)=n(A)+n(B)-n(A\cap B)+n((A\cup B)')$

$n(A\cup B)=n(A-B)+n(B-A)+n(A\cap B)$

$n(A-B)=n(A \cup B)-n(B)$

$n(A-B)=n(A)-n(A\cap B)$

$n(A')=n(U)-n(A)$

$n(PUQUR)=n(P)+n(Q)+n(R)-n(P\cap Q)-n(Q\cap R)-n(R\cap P)+n(P\cap Q\cap R)$

Operations on Sets

1. **Union**: The union of sets A and B is the set of elements that are in A, in B, or in both. It is denoted by $A \cup B$.

2. **Intersection**: The intersection of sets A and B is the set of elements that are in both A and B. It is denoted by $A \cap B$.

3. **Difference**: The difference of sets A and B is the set of elements that are in A but not in B. It is denoted by $A - B$.

4. **Complement**: The complement of a set A is the set of elements that are in the universal set U but not in A. It is denoted by A^c or $\overline{A}$.

Basic Concepts

1. **Set**: A collection of distinct objects, considered as an object in its own right. For example, a set of natural numbers, a set of letters in the alphabet, etc.

   - **Notation**: Sets are usually denoted by capital letters (e.g., A, B, C). The elements of a set are listed within curly braces (e.g., $A = \{1, 2, 3\}$).

2. **Element**: An object that belongs to a set. If a is an element of set A, it is written as $a \in A$. If b is not an element of set A, it is written as $b \notin A$.

3. **Subset**: A set A is a subset of set B if every element of A is also an element of B. This is written as $A \subseteq B$.

4. **Empty Set**: A set with no elements, denoted by $\emptyset$ or $\{\}$.

5. **Universal Set**: The set that contains all the objects under consideration, usually denoted by U.

Theorem of set

Set theory is rich with theorems that describe the relationships and properties of sets. Here are some fundamental theorems in set theory:

1. Subset Theorem

If A and B are sets, then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

2. De Morgan's Laws

For any two sets A and B:

- The complement of the union of A and B is the intersection of their complements:

$$(A \cup B)^c = A^c \cap B^c$$

- The complement of the intersection of A and B is the union of their complements:

$$(A \cap B)^c = A^c \cup B^c$$

3. Union and Intersection with Universal Set

For any set A and the universal set U:

- The union of A with the universal set U is U:

$$A \cup U = U$$

- The intersection of A with the universal set U is A:

$$A \cap U = A$$

4. Union and Intersection with Empty Set

For any set A:

- The union of A with the empty set $\emptyset$ is A:

$$A \cup \emptyset = A$$

- The intersection of A with the empty set $\emptyset$ is the empty set:

$$A \cap \emptyset = \emptyset$$

5. Idempotent Laws

For any set A:

- The union of A with itself is A:

$$A \cup A = A$$

- The intersection of A with itself is A:

$$A \cap A = A$$

6. Distributive Laws

For any sets A, B, and C:

- Union distributes over intersection:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

- Intersection distributes over union:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

7. Absorption Laws

For any sets A and B:

- Union with intersection:

$$A \cup (A \cap B) = A$$

- Intersection with union:

$$A \cap (A \cup B) = A$$

8. Complement Laws

For any set A:

- The complement of the complement of A is A:

$$(A^c)^c = A$$

- The union of A and its complement is the universal set:

$$A \cup A^c = U$$

- The intersection of A and its complement is the empty set:

$$A \cap A^c = \emptyset$$

9. **Double Complement Law**

For any set A:

- $(A^c)^c = A$

10. **Power Set Theorem**

For any set A, the power set of A, denoted $\mathcal{P}(A)$, is always larger than A itself if A is non-empty. Specifically, the cardinality of $\mathcal{P}(A)$ is $2^{|A|}$, where $|A|$ is the cardinality of A.

These theorems are foundational and widely used in mathematics, computer science, and logic to prove more complex results and to develop a deeper understanding of sets and their properties.

Conclusion:-

Only set theoretic conceptions can precisely define many mathematical topics. Graphs, manifolds, rings, vector spaces, and relational algebras are all examples of mathematical structures that can be characterized as sets satisfying various (axiomatic) qualities. In mathematics, equivalence and order relations are common, and set theory can be used to describe the theory of mathematical relations.

The idea of introducing the principles of naïve set theory early in mathematics education has gained appeal as set theory has gained acceptance as a foundation for modern mathematics.

Linear Algebra

Linear Algebra is a branch of mathematics that studies vectors, vector spaces (also called linear spaces), linear transformations, and systems of linear equations. It's a foundational subject with applications across science, engineering, computer science, economics, and more.

Key Concepts in Linear Algebra

Vectors and Vector Spaces:

Vectors: Objects that have both magnitude and direction. They can be represented as points in space or as an ordered list of numbers.

Vector Spaces: Collections of vectors that can be added together and multiplied by scalars (real numbers) to produce another vector in the same space. Examples include Euclidean space, function spaces, and spaces of matrices.

Matrices:

Rectangular arrays of numbers that can represent a system of linear equations, linear transformations, and more.

Matrix Operations: Includes addition, subtraction, multiplication, and finding inverses.

Systems of Linear Equations:

A collection of linear equations involving the same set of variables.

Methods for solving systems include substitution, elimination, and using matrix operations like Gaussian elimination.

Determinants and Inverses:

Determinant: A scalar value that can be computed from the elements of a square matrix and provides important properties of the matrix (like whether it is invertible).

Inverse Matrix: A matrix that, when multiplied by the original matrix, yields the identity matrix. Not all matrices have inverses.

Eigenvalues and Eigenvectors:

Eigenvectors: Vectors that, when a linear transformation is applied, change only in scale (not direction).

Eigenvalues: Scalars that represent how much the eigenvector is scaled during the transformation.

Linear Transformations:

Functions that map vectors to other vectors in a linear manner. They can be represented by matrices.

Matrices

Determinants and matrices are fundamental concepts in linear algebra. They are instrumental in solving linear equations using Cramer's rule, specifically for non-homogeneous equations in linear form. The determinant is applicable only for square matrices. If a matrix has a determinant of zero, it is referred to as a singular determinant . If the determinant is one, it is termed as unimodular . For a system of equations to have a unique solution, the determinant of the matrix should be nonzero, meaning it has to be nonsingular. In this article, we will explore the definition, types, and properties of determinants and matrices, along with examples for better understanding.

Matrix

Matrices are essentially ordered rectangular arrays of numbers, utilized to represent linear equations. They consist of rows and columns, and various mathematical operations such as addition, subtraction, and multiplication can be performed on them. If a matrix has 'm' rows and 'n' columns, it is represented as an m × n matrix.

$$A = \begin{bmatrix} m & n & o & p \\ q & r & s & t \\ u & v & w & x \\ y & z & a & b \end{bmatrix}$$

Now its determinant |A| is defined as

$$|A| = \begin{vmatrix} m & n & o & p \\ q & r & s & t \\ u & v & w & x \\ y & z & a & b \end{vmatrix}$$

$$= m\begin{vmatrix} r & s & t \\ v & w & x \\ z & a & b \end{vmatrix} - n\begin{vmatrix} q & s & t \\ u & w & x \\ y & a & b \end{vmatrix} + o\begin{vmatrix} q & r & t \\ u & v & x \\ y & z & b \end{vmatrix} - p\begin{vmatrix} q & r & s \\ u & v & w \\ y & z & a \end{vmatrix}$$

Various Types of Matrices

Matrices come in various types. Let’s look at some examples of different types of matrices.

If $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a Null Matrix

$If\ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a identity matrix

$$If\ A=\begin{bmatrix} 2 & 3 & -5 \\ 3 & 1 & 6 \\ -5 & 2 & 2 \end{bmatrix}$$ is a symmetric matrix

$$If\ A=\begin{bmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$ is a diagonal matrix

$$If\ A=\begin{bmatrix} 7 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 8 \end{bmatrix}$$ is a upper diagnaol matrix

$$If\ A=\begin{bmatrix} 7 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 8 \end{bmatrix}$$ is a lower diagonal matrix

Understanding the Inverse of a Matrix

The inverse of a matrix is typically defined for square matrices. For every m × n square matrix, an inverse matrix exists. If A is a square matrix, then A -1 is the inverse of matrix A and abides by the property:

AA -1 = A -1 A = I, where I is the Identity matrix.

Note that the determinant of the square matrix should not be zero.

Transpose of a Matrix

The transpose of a matrix is obtained by interchanging its rows and columns. If A is a matrix, then the transpose of the matrix is denoted by A T .

For instance, if we consider a 3×3 matrix, say A, then the transpose of A, i.e., A T is given by:

$$If\ A=\begin{bmatrix} 7 & 2 & 1 \\ 1 & 2 & 0 \\ 2 & 3 & 8 \end{bmatrix}$$

$$A^T=\begin{bmatrix} 7 & 1 & 2 \\ 2 & 2 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

If the given square matrix is a symmetric matrix, then the matrix A should be equal to A T .

This implies that A = A T .

Determinant

The determinant of a square matrix can be defined in various ways.

The simplest way to calculate the determinant is by considering the elements of the top row and their corresponding minors. You start by taking the first element of the top row and multiplying it by its minor, then subtract the product of the second element and its minor. Continue this process of alternately adding and subtracting the product of each element of the top row with its respective minor until all elements of the top row have been covered.

For instance, let's consider a 4×4 matrix A.

$$A = \begin{bmatrix} m & n & o & p \\ q & r & s & t \\ u & v & w & x \\ y & z & a & b \end{bmatrix}$$

Now its determinant |A| is defined as

$$|A| = \begin{vmatrix} m & n & o & p \\ q & r & s & t \\ u & v & w & x \\ y & z & a & b \end{vmatrix}$$

$$= m \begin{vmatrix} r & s & t \\ v & w & x \\ z & a & b \end{vmatrix} - n \begin{vmatrix} q & s & t \\ u & w & x \\ y & a & b \end{vmatrix} + o \begin{vmatrix} q & r & t \\ u & v & x \\ y & z & b \end{vmatrix} - p \begin{vmatrix} q & r & s \\ u & v & w \\ y & z & a \end{vmatrix}$$

Inverse of Matrix

Solve the following simultaneous equations for x and y

$3x+2y=5$

$7x+3y=10$

Solution:

$3x+2y=5$ $\qquad\qquad \begin{bmatrix}3 & 2\\7 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}5\\10\end{bmatrix}$

$7x+3y=10$

$Apply\ A^{-1}\times B$

$$A^{-1}=\frac{AdjA}{|A|}$$

$$|A|=\begin{bmatrix}3 & 2\\7 & 3\end{bmatrix}=9-14=-5$$

$$Adj\ A=\begin{bmatrix}3 & -2\\-7 & 3\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}\frac{3}{-5} & \frac{-2}{-5}\\ \frac{-7}{-5} & \frac{3}{-5}\end{bmatrix}$$

$X=A^{-1}\times B$

$$A^{-1}=\begin{bmatrix}\frac{3}{-5} & \frac{-2}{-5}\\ \frac{-7}{-5} & \frac{3}{-5}\end{bmatrix}\times\begin{bmatrix}5\\10\end{bmatrix}, \qquad A^{-1}=\begin{bmatrix}\frac{3}{-5}\times 5 & \frac{-2}{-5}\times 10\\ \frac{-7}{-5}\times 5 & \frac{3}{-5}\times 10\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}-3+4\\7-6\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

$x=1, y=1$

Simultaneous linear equations

Solve the following simultaneous equations for x, y and z:

$$x+y+z=6$$
$$x+2y+3z=14$$
$$x+4y+9z=36$$

The given set of equations can be written as AX = B

Solution:

$$A=\begin{bmatrix}1&1&1\\1&2&3\\1&4&9\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix} B=\begin{bmatrix}6\\14\\36\end{bmatrix}$$

$Apply\ A^{-1}\times B$

$$A^{-1}=\frac{AdjA}{|A|}$$

$$|A|=\begin{vmatrix}1&1&1\\1&2&3\\1&4&9\end{vmatrix}=1(18-12)-1(9-3)+1(4-2)$$

$$=1(6)-1(6)+1(2)$$
$$=6-6+2$$
$$|A|=2$$

Co-factor of A

$$A_{11}=\begin{vmatrix}2&3\\4&9\end{vmatrix}=18-12=6$$

$$A_{12}=-\begin{vmatrix}1&3\\1&9\end{vmatrix}=-(9-3)=-6$$

$$A_{13}=\begin{vmatrix}1&2\\1&4\end{vmatrix}=4-2=2$$

$$A_{21}=-\begin{vmatrix}1&1\\4&9\end{vmatrix}=-(9-4)=-5$$

$$A_{22}=\begin{vmatrix}1&1\\1&9\end{vmatrix}=(9-1)=8$$

$$A_{23}=\begin{vmatrix}1&1\\1&4\end{vmatrix}=-(4-1)=-3$$

$$A_{31}=\begin{vmatrix}1&1\\2&3\end{vmatrix}=3-2=1$$

$$A_{32}=-\begin{vmatrix}1&1\\1&3\end{vmatrix}=-(3-1)=-2$$

$$A_{33}=\begin{vmatrix}1&1\\1&2\end{vmatrix}=2-1=1$$

$$Co-Factor\ of\ A = \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$Adj\ A = \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$Apply\ X = A^{-1} \times B$$

$$= \begin{bmatrix} \frac{6}{2} & \frac{-5}{2} & \frac{1}{2} \\ \frac{-6}{2} & \frac{8}{2} & \frac{-3}{2} \\ \frac{1}{2} & \frac{-2}{2} & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} 6 \\ 14 \\ 36 \end{bmatrix}$$

Ans = $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Cramer's rule for solving system of linear equations

Here, one has to find out $\Delta, \Delta_x, \Delta_y, \Delta_z$. Δ is the determinant of the given matrix.
Δ_x is the determinant of matrix replacing the first column by the constant matrix
Δ_y is the determinant of matrix replacing the second column by the constant matrix
Δ_z is the determinant of matrix replacing the third column by the constant matrix

Solve the following equations by using Cramer's rule:

$$2x + 3y + 4z = 29$$
$$3x + 2y + 5z = 32$$
$$4x + 3y + 2z = 25$$

$$x = \frac{\Delta_x}{\Delta},\ y = \frac{\Delta_y}{\Delta},\ z = \frac{\Delta_z}{\Delta}$$

Solution:

$$\Delta = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 5 \\ 4 & 3 & 2 \end{vmatrix}$$

$$= 2(2\times 2)-(5\times 3)-3(3\times 2)-(5\times 4)+4(3\times 3)-(2\times 4)$$

$$= 2(4-15)-3(6-20)+4(9-8)$$

$$= 2(-11)-3(-14)+4(1)$$

$$= -22+42+4$$

$$\Delta = 24$$

$$\Delta_x = \begin{vmatrix} 29 & 3 & 4 \\ 32 & 2 & 5 \\ 25 & 3 & 2 \end{vmatrix}$$

$$= 29(2\times 2)-(5\times 3)-3(32\times 2)-(5\times 25)+4(32\times 3)-(2\times 25)$$

$$= 29(4-15)-3(64-125)+4(96-50)$$

$$= 29(-11)-3(-61)+4(46)$$

$$= -319+183+184$$

$$= -319+367$$

$$\Delta_x = 48$$

$$\Delta_y = \begin{vmatrix} 2 & 29 & 4 \\ 3 & 32 & 5 \\ 4 & 25 & 2 \end{vmatrix}$$

$$= 2(32\times 2)-(5\times 25)-29(3\times 2)-(5\times 4)+4(3\times 25)-(32\times 4)$$

$$= 2(64-125)-29(6-20)+4(150-128)$$

$$= 2(-61)-29(-14)+4(-53)$$

$$= -122+406+312$$

$$= -334+406$$

$$\Delta_y = 72$$

$$\Delta_z = \begin{vmatrix} 2 & 3 & 29 \\ 3 & 2 & 32 \\ 4 & 3 & 25 \end{vmatrix}$$

$$= 2(2\times 25)-(32\times 3)-3(3\times 25)-(32\times 4)+29(3\times 3)-(2\times 4)$$

$$= 2(50-96)-3(75-128)+29(9-8)$$

$$= 2(-46)-3(-53)+29(1)$$

$$= -92+159+29$$

$$= -92+188$$

$$\Delta_z = 96$$

$$x = \frac{\Delta_x}{\Delta} = \frac{48}{24} = 2$$

$$y = \frac{\Delta_y}{\Delta} = \frac{72}{24} = 3$$

$$z = \frac{\Delta_z}{\Delta} = \frac{96}{24} = 4$$

$$x = 2,\ y = 3,\ z = 4$$

Input-Output Analysis

Input-output is a novel technique invented by Professor Wassily W. Leontief in 1951. It is used to analyse inter-industry relationship in order to understand the inter-dependencies and complexities of the economy and thus the conditions for maintaining equilibrium between supply and demand.

Thus it is a technique to explain the general equilibrium of the economy. It is also known as “inter-industry analysis”. Before analysing the input-output method, let us understand the meaning of the terms, “input” and “output”. According to Professor J.R. Hicks, an input is “something which is bought for the enterprise” while an output is “something which is sold by it.”

An input is obtained but an output is produced. Thus input represents the expenditure of the firm, and output its receipts. The sum of the money values of inputs is the total cost of a firm and the sum of the money values of the output is its total revenue.

The input-output analysis tells us that there are industrial interrelationships and inter-dependen-cies in the economic system as a whole. The inputs of one industry are the outputs of another industry and vice versa, so that ultimately their mutual relationships lead to equilibrium between supply and demand in the economy as a whole.

Coal is an input for steel industry and steel is an input for coal industry, though both are the outputs of their respective industries. A major part of economic activity consists in producing intermediate goods (inputs) for further use in producing final goods (outputs). There are flows of goods in "whirlpools and cross currents" between different industries. The supply side consists of large inter-industry flows of intermediate products and the demand side of the final goods. In essence, the input-output analysis implies that in equilibrium, the money value of aggregate output of the whole economy must equal the sum of the money values of inter-industry inputs and the sum of the money values of inter-industry outputs.

1. Main Features:

The input-output analysis is the finest variant of general equilibrium. As such, it has three main elements; Firstly, the input-output analysis concentrates on an economy which is in equilibrium. Sec-ondly, it does not concern itself with the demand analysis. It deals exclusively with technical problems of production. Lastly, it is based on empirical investigation. The input-output analysis consists of two parts: the construction of the input-output table and the use of input-output model.

2. The Static Input-Output Model:

Assumptions:

This analysis is based on the following assumptions:

(i) The whole economy is divided into two sectors—"inter-industry sectors" and "final-demand sectors," both being capable of sub-sectoral division.

(ii) The total output of any inter-industry sector is generally capable of being used as inputs by other inter-industry sectors, by itself and by final demand sectors.

(iii) No two products are produced jointly. Each industry produces only one homogeneous product.

(iv) Prices, consumer demands and factor supplies are given.

(v) There are constant returns to scale.

(vi) There are no external economies and diseconomies of production.

(vii) The combinations of inputs are employed in rigidly fixed proportions. The inputs remain in constant proportion to the level of output. It implies that there is no substitution between different materials and no technological progress. There are fixed input coefficients of production.

The input-output model relates to the economy as a whole in a particular year. It shows the values of the flows of goods and services between different productive sectors especially inter-industry flows.

Explanation:

For understanding, a three-sector economy is taken in which there are two inter-industry sec-tors, agriculture and industry, and one final demand sector.

Table 1 provides a simplified picture of such economy in which the total output of the industrial, agricultural and household sectors is set in rows (to be read horizontally) and has been divided into the agricultural, industrial and final demand sectors. The inputs of these sectors are set in columns. The first row total shows that altogether the agricultural output is valued at Rs. 300 crores per year.

Table 1 : Input-Output Table

(In value terms) (Rs. Crores)

		Purchasing Sectors			
	Sectors	*1 Inputs to Agriculture*	*2 Inputs to Industry*	*3 Final Demand*	*Total Output or Total Revenue*
Selling Sectors	Agriculture	50	150	100	300
	Industry	100	250	150	500
	Value added*	150	100	0	250
	Total input or Total Cost	300	500	250	1050

Of this total, Rs. 100 crores go directly to final consumption (demand), that is, household and government, as shown in the third column of the first row. The remaining output from agriculture goes as inputs: 50 to itself and 150 to industry. Similarly, the second row shows the distribution of total output of the industrial sector valued at Rs. 500 crores per year. Columns 1, 2 and 3 show that 100 units of manufactured goods go as inputs to agriculture, 250 to industry itself and 150 for final con-sumption to the household sector.

Let us take the columns (to be read downwards). The first column describes the input or cost structure of the agricultural industry. Agricultural output valued at Rs. 300 crores is produced with the use of agricultural goods worth Rs. 50, manufactured goods worth Rs. 100 and labour or/and manage-ment services valued at Rs. 150. To put it differently, it costs Rs. 300 crores to get revenue of Rs. 300 crores from the agricultural sector. Similarly, the second column explains the input structure of the industrial sector (i.e., 150 + 250 + 100 = 500).

Thus "a column gives one point on the production function of the corresponding industry." The 'final demand' column shows what is available for consumption and government expenditure. The third row corresponding to this column has been shown as zero. This means that the household sector is simply a spending (consuming) sector that does not sell anything to itself. In other words, labour is not directly consumed.

There are two types of relationships which indicate and determine the manner in which an economy behaves and assumes a certain pattern of flows of resources.

They are:

(a) The internal stability or balance of each sector of the economy, and

(b) The external stability of each sector or inter-sectoral relationships. Professor Leontief calls them the "fundamental relationships of balance and structure." When expressed mathematically they are known as the "balance equations' and the "struc-tural equations".

If the total output of say X. of the 'ith' industry is divided into various numbers of industries 1, 2, 3, n, then we have the balance equation:

$X1 = xi1 + xi2 + xi3 + xin\ldots\ldots + D1$

and if the amount say У. absorbed by the "outside sector" is also taken into consideration, the balance equation of the ith industry becomes

It is to be noted that Yi stands for the sum of the flows of the products of the ith industry to consump-tion, investment and exports net of imports, etc. It is also called the "final bill of goods" which it is the function of the output to fill. The balance equation shows the conditions of equilibrium between de-mand and supply. It shows the flows of outputs and inputs to and from one industry to other industries and vice versa.

Since x12 stands for the amount absorbed by industry 2 of the ith industry, it follows that xij stands for the amount absorbed by the ith industry of jth industry.

The "technical coefficient" or "input coefficient" of the ith industry is denoted by:

$aij = xij/Xj$

where xij is the flow from industry i to industry j, Xj is the total output of industry aij and aij, as already noted above, is a constant, called "technical coefficient" or "flow coefficient" in the ith industry. The technical coefficient shows the number of units of one industry's output that are required to produce one unit to another industry's output.

Equation (3) is called a "structural equation." The structural equation tells us that the output of one industry is absorbed by all industries so that the flow structure of the entire economy is revealed. A number of structural equations give a summary description of the economy's existing technological conditions.

Using equation (3) to calculate the aij for our example of the two-sector input-output Table 1, we get the following technology matrix.

Table 2: Technology Coefficient Matrix A

	Agriculture	*Industry*
Agriculture	50/300= .17	150/500= .30
Industry	100/300= .33	250/500= .50

These input coefficients have been arrived at by dividing each item in the first column of Table 1 by first row total, and each item in the second column by the second row, and so on. Each column of the technological matrix reveals how much agricultural and industrial sectors require from each other to produce a rupee's worth of output. The first column shows that a rupee's worth of agricultural output requires inputs worth 33 paise from industries and worth 17 paise from agriculture itself.

Hawkins - Simon condition open and closed models

1. The Hawkins – Simon conditions

Hawkins – Simon conditions ensure the viability of the system.

• If B is the technology matrix Hawkins – Simon conditions are

i. the main diagonal elements in I – B must be positive and

ii. |I – B| must be positive.

Types

1. The Open Model

If, besides the n industries, the model contains an "open" sector (say, households) which exogenously determines a final demand (non-input demand) for the product of each industry and which supplies a primary input (say, labour service) not produced by the n industries themselves, then the model is an open one

2. The Closed Model

If the exogenous sector of the open input-output model is absorbed into the system as just another industry, the model will become a closed one. In such a model, final demand and primary input do not appear; in their place will be the input requirements and the output of the newly conceived industry. All goods will now be intermediate in nature, because everything is produced only for the sake of satisfying the input requirements of the (n + 1) sectors in the model.

UNIT II

DIFFERENTIAL CALCULUS

LIMIT AND CONTINUITY

The concept of Limit is the most basic concept in Calculus. A particular case of limit leads to the notion of Continuity. In this chapter, we shall expalin the elementary knowledge of these two concepts.

LIMIT

Let 'x' be a real variable and 'a' be a finite real number. We often say "$x \to a$" (x tends to a). This $x \to a$ is associted with the approach of the successive values of the variable 'x' towards a given number 'a'.

Limiting value of a function

What value does one variable 'y' approach when another variable 'x' approaches a specific finite quantity 'a' ? This question makes proper sense only if y is a function of x.

Suppose, $y = f(x)$. Here, our problem is to what value y {(i.e., f(x)} approaches when 'x' approaches 'a' ? We say that y {i.e., f(x)} has a limit 'l' as $x \to a$. We describe this situation by writing

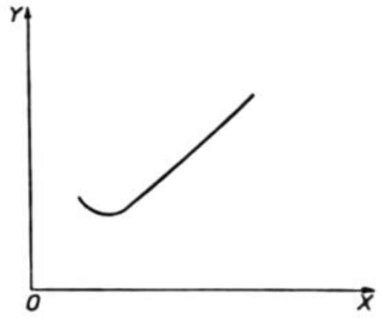
$$\lim_{x \to a} f(x) = l \quad \text{or} \quad \underset{x \to a}{\mathrm{Lt}}\, f(x) = l$$

Read: Limiting value of y {i.e., f(x)} when x tends to a is 'l'.

CONTINUITY OF A FUNCTION

The word 'Continuity' is used to indicate the absence of any gap. If there is no break of the curve of a function, we say that the function is continuous. If there is a break of the curve of a function, we say that the function is discontinuous. The Figures 2.27 and 2.28 illustrate the continuity and discontinuity of the function.

Continuity of a Function



Continuity of a Function

Discontinuity of a Function

Discontinuity of a Function

We may say that a function y = f(x) is continuous, if small change in the values of independent variable x cause small change in the values of the dependent variable y and the function is said to be discontinuous if small change in x can make a sudden change (or jump) in the value of y.

Examples:

Find $\underset{x \to 2}{\text{Lt}} \frac{x^5 - 22}{x^2 - 2}$

Solution:

$$\underset{x \to 2}{\text{Lt}} \frac{x^5 - 22}{x^2 - 2} = \frac{\underset{x \to 2}{\text{Lt}} (x^5 - 22)}{\underset{x \to 2}{\text{Lt}} (x^2 - 2)}$$

$$= \frac{32 - 22}{4 - 2} = \frac{10}{2} = 5.$$

Find $\underset{x \to 2}{\text{Lt}} \frac{(5x^2 - 16)}{\sqrt{3x^2 + 4}}$.

Solution:

$$\underset{x \to 2}{\text{Lt}} \frac{5x^2 - 16}{\sqrt{3x^2 + 4}}$$

$$= \frac{\underset{x \to 2}{\text{Lt}} (5x^2 - 16)}{\underset{x \to 2}{\text{Lt}} \sqrt{3x^2 + 4}}$$

$$= \frac{20 - 16}{\sqrt{12 + 4}}$$

$$= \frac{4}{\sqrt{16}}$$

$$= \frac{4}{4} = 1.$$

A. Maxima

A function f(x) is said to have attained its "Maximum value" or "Maxima" at x = a, if the function stops to increase and begins to decrease at x = a. In other words, $f(x_1)$ is a maximum value of a function 'f', if it is the highest of all its values for values of x in some neighbourhood of A (Fig. 3.1).

B. Minima

A function f(x) is said to have attained its "Minimum value" or "Minima" at x = b, if the function stops to decrease and begins to increase at x = b. In other words, $f(x_2)$ is a minimum value of a function 'f' if it is the lowest of all its values for values of x in some neighbourhood of B (Fig 3.1).

The Maxima and Minima of the function are called the "Extreme Values" of the function.

Maxima And Minima Of One Variable

Let us consider a function Y = f(x). If we plot this function, the function takes the form as given in the Figure 3.1. We consider three points A, B, and C, where $\frac{dY}{dx} = 0$ in each case. That is in all stationary level the derivative is zero.

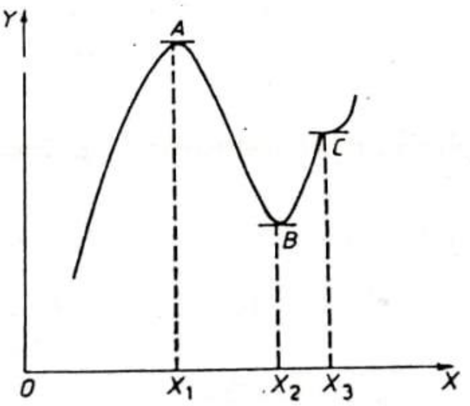


Fig. 3.1 : Maxima and Minima

(1) At Point 'A'

We call the point 'A' a maximum point because Y has a maximum value at this point when $X = OX_1$. Therefore, we say that at point A,Y is maximum. It means that the value of Y at A is higher than any value on either side of A and also on the left side of point 'A', the curve is increasing and decreasing on the right side of 'A'.This also means that value of Y increases with the increase in X up to the point A. But it must fall after point A has been reached. As value of X increases from X_1 to X_2 the slope of the curve is decreasing or the slope of the curve changes from zero to negative values (since $\frac{dY}{dx} = 0$ at point 'A'). Thus, if 'A' is to be maximum,

i) $\frac{dY}{dx} = 0$ and also

ii) $\frac{d^2Y}{dx^2} < 0$ i.e., –ve at point 'A'.

(2) At Point 'B'

We call the point 'B' a minimum point because Y has a minimum value when $X = OX_2$. Therefore, we say that at point 'B', Y is minimum. It means that the value of Y at 'B' is lower than any value on either side of B and also on the left side of point 'B', the curve is decrcasing and increasing on the right side of 'B'. This also means that value of Y decreases with the increase in X upto the point 'B'. But, it must increase after point 'B' has been reached. As value of X increases form X_2 to X_3, the slope of the curve is increasing or the slope of the curve changes from zero to positive values (since $\frac{dY}{dx} = 0$ at point 'B').

Thus if 'B' is to be minimum,

i) $\frac{dY}{dx} = 0$ and also

ii) $\frac{d^2Y}{dx^2} > 0$ i.e., ; +ve at point 'B'.

Table 3.2 : Conditions for Maxima and Minima

	Maxima	Minima
1. First Order Condition (Necessary Condition)	$f'(x)$ or $\frac{dY}{dx} = 0$	$f'(x)$ or $\frac{dY}{dx} = 0$
2. Second Order Condition (Sufficient Condition)	$f''(x)$ or $\frac{d^2Y}{dx^2} < 0$ or –ve	$f''(x)$ or $\frac{d^2Y}{dx^2} > 0$ or +ve

(3) At Point 'C'

Point of Inflexion is a point at which a curve is changing from concave upward to concave downward, or vice versa. In the Figure 3.1,we call this point 'C' a"Point of Inflection" or "Inflexional Point". Because on either sides of point 'C' the curve slopes upwards. Therefore, on either sides of 'C', the first order derivative is greater than zero i.e., positive except at point C. Hence, this point is called "Inflexional Point" or "Point of Inflexion", because of mere bend in the curve. At the inflexional points, the second order derivative is equal to zero. "Inflexional points may be stationary and inflexional. In this case, both first and second order derivatives are zero. If the point is inflexional and non statioary, the first order derivative is not equal to zero. But the second order derivative is equal to zero.

Thus, if 'C' is to be "Inflexional point", then

i) $\frac{dY}{dx} \geq 0$ and also ii) $\frac{d^2Y}{dx^2} = 0.$

Examples:

1. Given the function $y = x^3 - 3x^2 + 7$, find the point of inflexion.

Solution:

$$\frac{dy}{dx} = 3x^2 - 6x$$

The first condition for inflexion is $\frac{dy}{dx} = 0$

Therefore, $3x^2 - 6x = 0$

$$x\ (3x - 6) = 0$$

i) $x = 0$ or ii) $3x - 6 = 0$

$$3x = 6$$

$$x = \frac{6}{3} = 2.$$

Thus, $\frac{dy}{dx} = 0$, when $x = 0$ or $x = 2$.

a) If $x = 0$, $y = (0)^3 - 3\ (0)^2 + 7$

$y = 7$

b) If $x = 2$, $y = (2)^3 - 3\ (2)^2 + 7$

$y = 8 - 12 + 7 = 3$

The second condition for inflexion is $\frac{d^2y}{dx^2} = 0$

$$\frac{d^2y}{dx^2} = 6x - 6$$

$$6x - 6 = 0$$

$$6x = 6$$

$$x = \frac{6}{6} = 1 > 0.$$

The point of inflexion is $x = 0$ and $y = 7$ or $x = 2$ and $y = 3$.

Find the maxima and minima of the function $y = 2x^3 - 6x$.

Solution:

$$y = 2x^3 - 6x$$

$$\frac{dy}{dx} = 6x^2 - 6$$

At the maximum or minimum $\frac{dy}{dx} = 0$

Therefore, $6x^2 - 6 = 0$

$$6x^2 = 6$$

$$x^2 = \frac{6}{6}$$

$$x^2 = 1$$

$$x = \pm 1$$

$x = -1$ and $x = 1$ give maximum or minimum

$$\frac{dy}{dx} = 6x^2 - 6x$$

$$\frac{d^2y}{dx^2} = 12x - 6$$

When $x = -1$, $\frac{d^2y}{dx^2} = -18 < 0$ i.e., negative

When $x = 1$, $\frac{d^2y}{dx^2} = 6 > 0$ i.e., positive.

Therefore, **$x = -1$ gives the maximum value of the function and $x = 1$ gives the minimum value of the function.**

Derivatives

So far we concerned ourselves with functions of one independent variable: for example, while introducing the technique of differentiation, we thought of total utility y as a function of consumption of one commodity x, so that y=f(x). Also, all our rules of differentiation assumed y as function of only one variable x.

But the total utility or any quantity may in fact be function of two or more independent variables. We can cite many such examples in economics. If z tones of wheat are produced on y acres of land with x number of labourers; z is the function of x and y both, that is: z=f(x, y). If consumer purchases meat x and bread y for his lunch, then his total utility u will depend on amounts of both x and y consumed by him so that: u=f(x, y). Similarly, we can think of total utility u being function of amounts of different commodities $x_1, x_2, x_3 \ldots\ldots x_n$ consumed by the consumer in a form of function:

$$U=f(x_1, x_2, x_3 \ldots\ldots x_n)$$

Where $x_1, x_2, x_3 \ldots x_n$ is all independent of one another and that each can vary by itself without affecting others.

We consider a simple case of ‘U’ being dependent on two variables:

$$U=f(x, y)$$

If the variable x undergoes change while y remains constant, there will be a corresponding change in U, say it is Δ U, then:

$$U + \Delta U = f(x + \Delta x, y)$$
$$\frac{\Delta U}{\Delta x} = \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

f_z

Now we take limit of $\frac{\Delta U}{\Delta x}$ as Δ x tends to zero to find the derivatives:

We call this specific derivative as 'Partial Derivative' of U with respect to x. we call it "Partial" derivative to indicate that y in the function has beer, held constant.

Such partial derivative is assigned different symbol to indicate that this is the partial derivative with respect to x and that other variable y has been regarded as fixed. In place of letter'd' we use symbol '∂' and write partial derivative of U with respect to x as:

$$\frac{\partial U}{\partial x}\left(notas \frac{dU}{dx} \right)$$

Thus, $\underset{\Delta x \to 0}{Lt} \frac{\Delta U}{\Delta x} = \frac{\partial U}{\partial x} = \underset{\Delta x \to 0}{Lt} \frac{f(x = \Delta x, y) - f(x, y)}{\Delta x}$

Similarly, we can have the partial derivative of U with respect to y as:

$$\underset{\Delta x \to 0}{Lt} \frac{\Delta U}{\Delta y} = \frac{\partial U}{\partial y} = \underset{\Delta x \to 0}{Lt} \frac{f(x = \Delta y, y) - f(x, y)}{\Delta y}$$

Partial derivatives are also depicted by symbols as $f_{x,} f_{y,} f_{z,\ldots\ldots}$, where subscript indicates which independent variable (alone) is being allowed to vary. For example, $\frac{\Delta U}{\Delta x}$ can also be represented by f_z and $\frac{\partial U}{\partial y}$ by f_y.

Technique of Partial Differentiation

The process of taking partial derivative is called partial differentiation and it differs from previously discussed differentiation primarily in that we hold and treat all the independent variables constant except the one which is assumed to vary.

The practical technique of partial differentiation is illustrated by the following examples:

Ex. 1. Given function is U=5x-6y+8 and we are required to find partial derivatives. There can be only two partial derivatives.

(1) U with respect to x when y is held constant

$$\frac{\partial U}{\partial y} \text{ Or } f_x$$

(2) U with respect to y when x is held constant

$$\frac{\partial U}{\partial y} \text{ Or } f_y$$

In case of (1), since y is held constant it is treated as a constant term during differentiation; while in case of (2), since x is held constant it is treated as a constant term during differentiation; thus we have

(1) $\frac{\partial U}{\partial y}$ Or f_x =5-0+0=5(-6y and +8 are treated as constant terms)

(2) $\frac{\partial U}{\partial y}$ Or f_y =0-6+0=-6(5x and +8 are treated as constant terms)

Similarly if Z=ax+by+c then

$$f_z\left(=\frac{\partial U}{\partial z}\right)\frac{dy}{dx}\ \frac{\partial Z}{\partial x} = \text{a and } \frac{\partial Z}{\partial y} = \text{b.}$$

Partial Derivatives of Second Order

We can also find the partial derivatives of second or higher orders. The process of partial derivation can be repeated till the partial in the original function. For example:

(1) If $\frac{\partial Z}{\partial x}$(the partial derivative of Z with respect to x)happens to be the function of x and y, it could be differentiated partially with respect to x and written as

$$\frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial x}\right)$$

And denoted by $\frac{\partial^2 Z}{\partial x^2}$ or f_{xx} .

In other words, f_{xx} $\left(=\frac{\partial^2 Z}{\partial x^2}\right)$ is the second partial derivative obtained by partial derivation; first with respect to x and then again with respect to x.

Cross Partial Derivatives

Second order partial derivatives: f_{xy} and f_{yx} are known as cross partial derivatives. This may be noticed that the cross partial derivatives

$$f_{xy}\left(\frac{\partial^2 Z}{\partial x \partial y}\right) \text{ And } f_{yx}\left(\frac{\partial^2 Z}{\partial y \partial x}\right)$$

Are different only in the order in which Z has been differentiated partially:

(1) $\left(\frac{\partial^2 Z}{\partial y \partial x}\right)$Or f_{yx} indicates that the given function Z is first partially differentiated with respect to x and then with respect to y.

(2) $\left(\frac{\partial^2 Z}{\partial x \partial y}\right)$ Or f_{xy} indicates that given function Z is first partially differentiated with respect to y and then with respect to x.

However, it can be shown that under certain conditions the cross partial derivatives are identical in value,i.e.,

$$\left(\frac{\partial^2 Z}{\partial y \partial x}=\frac{\partial^2 Z}{\partial x \partial y}\right); \text{ or } f_{yx}=f_{xy}$$

In other words, the order of the partial derivation does not make any difference in the process of partial differentiation.

We take a few examples to show this property of partial differentiation.

$$\text{Ex.} \quad Z = \frac{x+4}{2x+5y}$$

$$\frac{\partial Z}{\partial x} = f_x = \frac{5y-8}{\left(2x+5y\right)^z}$$

Since $\frac{5y-8}{\left(2x+5y\right)^z}$ is the function of x and y both. We can find second order derivatives as follows:

$$\frac{\partial^2 Z}{\partial x \partial x} = f_{zx} = \frac{0\left(2x+5y\right)^2 - 2\left(2x+5y)(2)(5y-8\right)}{\left(2x+5y\right)^4} = \frac{-4\left(5y-8\right)}{\left(2x+5y\right)^3}$$

$$\frac{\partial^2 Z}{\partial y \partial x} = f_{yx} = \frac{5\left(2x+5y\right)^2 - 2\left(2x+5y)(5)(5y-8\right)}{\left(2x+5y\right)^4} = \frac{10x-25y+80}{\left(2x+5y\right)^3}$$

And

$$\frac{\partial Z}{\partial y} = f_y = \frac{-5x-20}{\left(2x+5y\right)^z}$$

$$\frac{\partial^2 Z}{\partial x \partial y} = f_{zy} = \frac{-5\left(2x+5y\right)^2 - 2\left(2x+5y)(2)(5y-20\right)}{\left(2x+5y\right)^4} = \frac{10x-25y+80}{\left(2x+5y\right)^3}$$

Hence $f_{zy} = f_{yz}$

$$\frac{\partial^2 Z}{\partial y \partial y} = f_{yy} = \frac{-0\left(2x+5y\right)^2 - 2\left(2x+5y)(5)(-5x-20\right)}{\left(2x+5y\right)^4} = \frac{50\left(x+4\right)}{\left(2x+5y\right)^3}$$

Partial Derivatives of Functions of More Than Two Variables

When defining the partial derivatives we considers a simple case of functions involving only two variables, e.g., Z=f(x, y). But we can have a function involving more than two variables; [U=f(x, y, z)]. Partial derivatives in this case may by define as:

(1) $f_x\left(=\frac{\partial U}{\partial x}\right)$ Which represents the change in U with respect to x when y and z are being held as constant?

(2) $f_y\left(=\frac{\partial U}{\partial y}\right)$ Represents the rate of change in U with respect to y when x and z are being held as constant.

(3) $f_z\left(=\frac{\partial U}{\partial z}\right)$ Represent the rate of change in U with respect to z when x and y are being held as constant.

The second order partial derivatives can be obtained as in the case of functions involving two variables.

Application of Partial Derivatives in Economics

In the case of a function of one variable y=f(x) we observed that the derivative $\frac{dy}{dx}$ depicts the rate of charge in y due to marginal change in x. similarly, partial derivatives $\frac{\partial U}{\partial x}$ in a function U=f(x, y, z) depicts the rate of change in U with the marginal change in x when y and z do not change. $\frac{\partial U}{\partial y}$ And $\frac{\partial U}{\partial z}$ gives the rate of change in U with respect to marginal change in y and z respectively when the remaining two variables are assumed to be held constants.

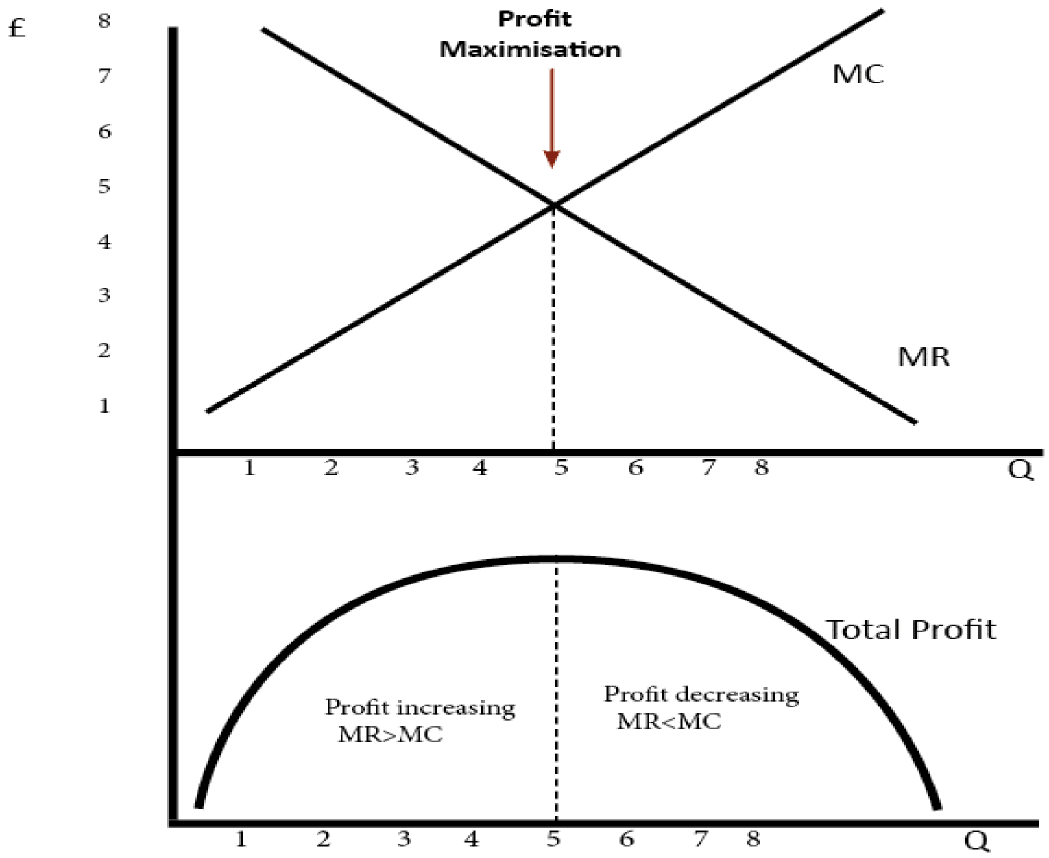
Profit & revenue maximization under perfect competition under monopoly

An assumption in classical economics is that firms seek to maximise profits.

Profit = Total Revenue (TR) – Total Costs (TC).

Therefore, profit maximisation occurs at the biggest gap between total revenue and total costs.

A firm can maximise profits if it produces at an output where marginal revenue (MR) = marginal cost (MC)



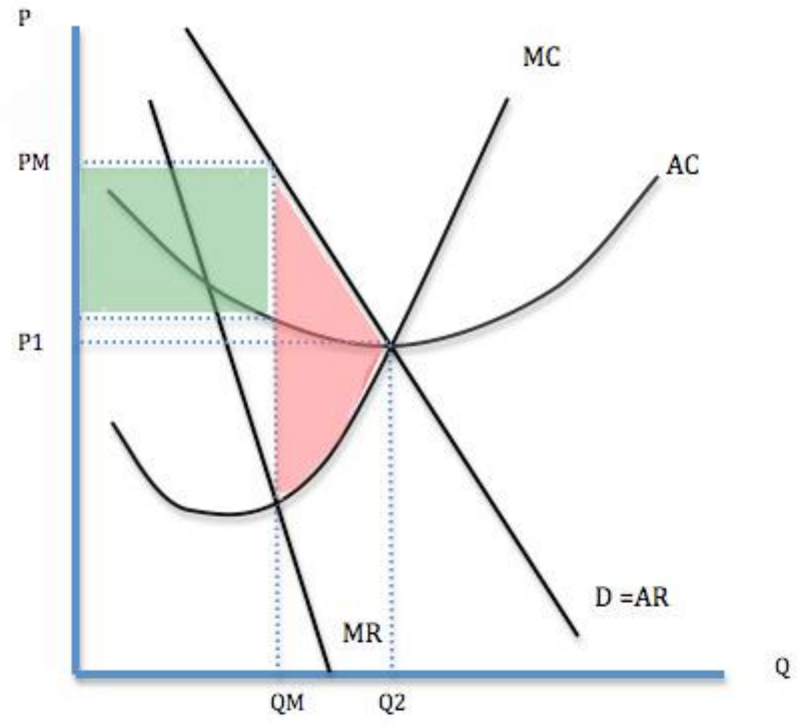
To understand this principle look at the above diagram.

If the firm produces less than Output of 5, MR is greater than MC. Therefore, for this extra output, the firm is gaining more revenue than it is paying in costs, and total profit will increase.

At an output of 4, MR is only just greater than MC; therefore, there is only a small increase in profit, but profit is still rising.

However, after the output of 5, the marginal cost of the output is greater than the marginal revenue. This means the firm will see a fall in its profit level because the cost of these extra units is greater than revenue.

Profit maximization under monopoly

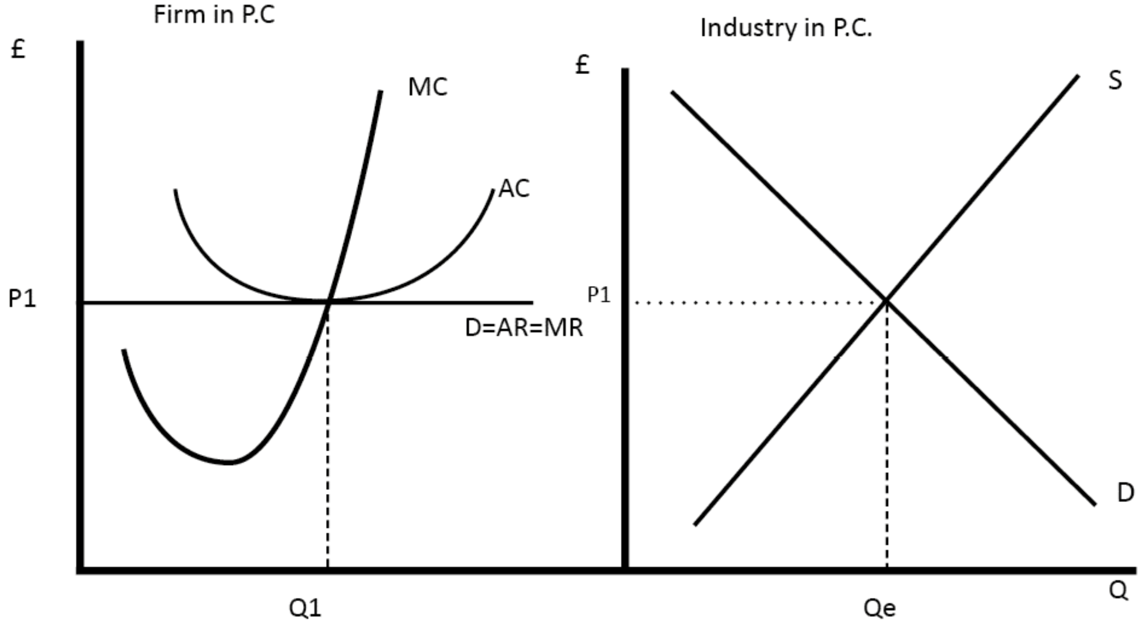


In this diagram, the monopoly maximises profit where MR=MC – at Qm. This enables the firm to make supernormal profits (green area). Note, the firm could produce more and still make normal profit. But, to maximise profit, it involves setting a higher price and lower quantity than a competitive market.

Note, the firm could produce more and still make a normal profit. But, to maximise profit, it involves setting a higher price and lower quantity than a competitive market.

Therefore, in a monopoly profit maximisation involves selling a lower quantity and at a higher price.

Profit maximization under Perfect Competition



In perfect competition, the same rule for profit maximisation still applies. The firm maximises profit where MR=MC (at Q1).

For a firm in perfect competition, demand is perfectly elastic, therefore MR=AR=D.

This gives a firm normal profit because at Q1, AR=AC.

Maximizing excise tax revenue in monopolistic competitive market

In a monopolistically competitive market, the rule for maximizing profit is to set MR = MC—and price is higher than marginal revenue, not equal to it because the demand curve is downward sloping.

Profit is maximized where marginal revenue is equal to marginal cost. In this case, for a competitive firm, marginal revenue is equal to price. So profit is maximized where price is equal to marginal cost or at this point right here.

Minimization of cost

Cost minimisation is a financial strategy that aims to achieve the most cost-effective way of delivering goods and services to the require level of quality. It is important to remember that cost minimisation is not about reducing quality or short-changing customers – it always remains important to meet customer needs.

Cost Minimisation for a Given Output:

In the theory of production, the profit maximisation firm is in equilibrium when, given the cost- price function, it maximises its profits on the basis of the least cost combination of factors. For this, it will choose that combination which minimises its cost of production for a given output. This will be the optimal combination for it.

Assumptions:

This analysis is based on the following assumptions:

1. There are two factors, labour and capital.
2. All units of labour and capital are homogeneous.
3. The prices of units of labour (w) and that of capital (r) are given and constant.
4. The cost outlay is given.
5. The firm produces a single product.
6. The price of the product is given and constant.
7. The firm aims at profit maximisation.
8. There is perfect competition in the factor market.

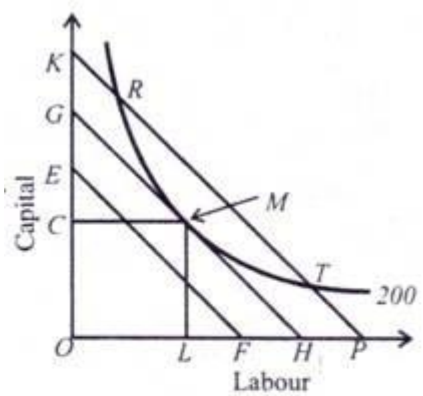


Fig. 15

Explanation:

Given these assumptions, the point of least-cost combination of factors for a given level of output is where the isoquant curve is tangent to an isocost line. In Figure 15, the isocost line GH is tangent to the isoquant 200 at point M. The firm employs the combination of OC of capital and OL of labour to produce 200 units of output at point M with the given cost-outlay GH.

At this point, the firm is minimising its cost for producing 200 units. Any other combination on the isoquant 200, such as R or T, is on the higher isocost line KP which shows higher cost of production. The isocost line EF shows lower cost but output 200 cannot be attained with it. Therefore, the firm will choose the minimum cost point M which is the least-cost factor combination for producing 200 units of output. M is thus the optimal combination for the firm.

The point of tangency between the isocost line and the isoquant is an important first order condition but not a necessary condition for the producer's equilibrium.

There are two essential K or second order conditions for the equilibrium of the firm:

1. The first condition is that the slope of the isocost line must equal the slope of the isoquant curve. The slope of the isocost line is equal to the ratio of the price of labour (w) and the price of capital (r). The slope of the isoquant curve is equal to the marginal rate of technical substitution of labour and capital (MRTSLK) which is, in turn, equal to the ratio of the marginal product of labour to the marginal product of capital (MPL/MPK' condition for optimality can be written as.

w/r MPL/MPK = MRTSLK

The second condition is that at the point of tangency, the isoquant curve must be convex to the origin. In other words, the marginal rate of technical substitution of labour for capital (MRTSLK) must be diminishing at the point of tangency for equilibrium to be stable. In Figure 16, S cannot be the point of equilibrium for the isoquant IQ1 is concave where it is tangent to the isocost line GH. At point S, the marginal rate of technical substitution between the two factors increases if move to the right or left on the curve IQ1.

Moreover, the same output level can be produced at a lower cost AB or EF and there will be a comer solution either at C or F. If it decides to produce at EF cost, it can produce the entire output with only OF labour. If, on the other hand, it decides to produce at a still lower cost CD, the entire output can be produced with only OC capital.

Both the situations are impossibilities because nothing can be produced either with only labour or only capital. Therefore, the firm can produce the same level of output at point M, where the isoquant curve IQ is convex to the origin and is tangent to the isocost line GH. The analysis assumes that both the isoquants represent equal level of output, IQ = IQ1.

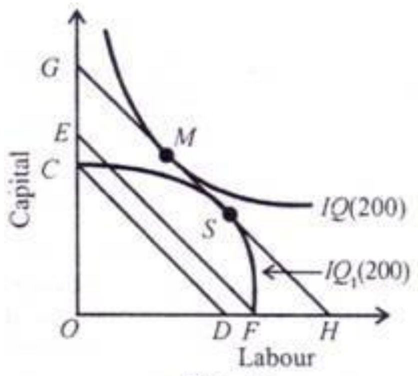


Fig. 16

Output-Maximisation for a Given Cost:

The firm also maximises its profits by maximising its output, given its cost outlay and the prices of the two factors. This analysis is based on the same assumptions, as given above. The conditions for the equilibrium of the firm are the same, as discussed above.

1. The firm is in equilibrium at point P where the isoquant curve 200 is tangent to the isocost line CL in Figure 17. At this point, the firm is maximising its output level of 200 units by employing the optimal combination of OM of capital and ON of labour, given its cost outlay CL.

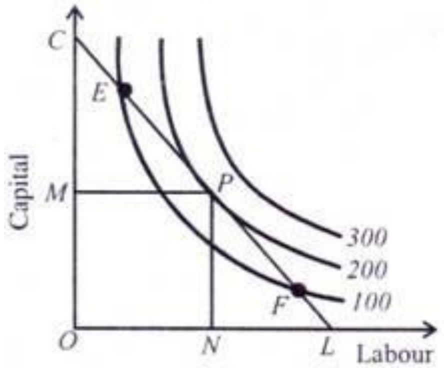


Fig. 17

But it cannot be at points E or F on the isocost line CL, since both points give a smaller quantity of output, being on the isoquant 100, than on the isoquant 200. The firm can reach the optimal factor com-bination level of maximum output by moving along the isocost line CL from either point E or F to point P.

This movement involves no extra cost because the firm re-mains on the same isocost line. The firm cannot attain a higher level of output such as isoquant 300 because of the cost constraint. Thus the equilibrium point has to be P with optimal factor combination OM+ ON. At point P, the slope of the isoquant curve 200 is equal to the slope of the isocost line CL. It implies w/r = MPL/MPK = MRTSLK.

2. The second condition is that the isoquant curve must be convex to the origin at the point of tangency with the isocost line, as explained above in terms of Figure 16.

UNIT III

OPTIMIZATION TECHNIQUES WITH CONSTRAINTS

Functions of Several Variables

Functions of several variables, also known as multivariable functions, are mathematical expressions that involve multiple input variables. These functions have important applications in various fields such as physics, engineering, and economics. In this essay, we will discuss the basic concepts of functions of several variables, including their definition, notation, and basic properties. We will also explore some of the main types of multivariable functions and their uses.

Definition and Notation

A function of several variables, or multivariable function, is a mathematical expression that depends on two or more input variables. These functions are represented by a single output variable, also known as the dependent variable. The notation for a function of several variables is typically written as $f(x,y,z)$, where x, y, and z represent the input variables and f represents the output variable. In this notation, the input variables are enclosed in parentheses, indicating that they are the inputs for the function.

Basic Properties

Like single-variable functions, multivariable functions have certain basic properties that are important to understand. One of the most important properties of multivariable functions is their domain, which is the set of all input values for which the function is defined. Another important property is the range, which is the set of all possible output values.

In addition to domain and range, multivariable functions also have partial derivatives. These are similar to the derivative of a single-variable function, but they measure the rate of change of the output variable with respect to each individual input variable. The partial derivatives are represented by symbols such as $\partial f/\partial x$ and $\partial f/\partial y$.

Types of Multivariable Functions

There are several different types of multivariable functions, each with its own unique properties and applications. Some of the most common types of functions include:

● Linear functions:

Linear functions are functions of the form $f(x,y) = a + bx + cy$, where a, b, and c are constants. These functions are important in physics and engineering because they represent straight lines in two-dimensional space.

● Quadratic functions:

Quadratic functions are functions of the form $f(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$, where a, b, c, d, e, and f are constants. These functions are important in mathematics and physics because they represent parabolas in two-dimensional space.

● Polar functions:

Polar functions are functions of the form $f(r,\theta) = r^2 + \theta$, where r and θ are input variables representing distance and angle, respectively. These functions are important in physics and engineering because they represent points in two-dimensional space using polar coordinates.

● Vector functions:

Vector functions are functions of the form $f(x,y) = (ax + by + c, dx + ey + f)$, where a, b, c, d, e, and f are constants. These functions are important in physics and engineering because they represent vectors in two-dimensional space.

Applications

Functions of several variables have many important applications in various fields. In physics, these functions are used to describe the motion of particles and the behavior of fluids. In engineering, they are used to design and optimize mechanical systems, such as gears and engines. In economics, they are used to analyze the relationship between different economic variables, such as supply and demand. In addition, functions of several variables are also used in machine learning, where they can be used to model complex datasets with multiple input variables. Furthermore, multivariable functions

1. A function of several variables is a mathematical rule that assigns a unique output to each set of input values.
2. These functions are typically represented using multiple variables, such as x, y, and z.
3. Functions of several variables can be used to model real-world situations, such as the relationship between temperature and humidity.
4. These functions can be graphed in multiple dimensions, such as on a 3D graph.
5. They can also be represented using equations, such as f(x, y) = x^2 + y^2.
6. Functions of several variables can be used to calculate partial derivatives, which describe how the function changes as one variable changes while the others remain constant.
7. These functions can also be used to calculate gradients, which describe the direction of the greatest rate of change.
8. They can be used to find extrema, such as maxima and minima, by using partial derivatives to find critical points
9. Functions of several variables can also be used to calculate multiple integrals, which can be used to calculate areas and volumes.1
0. These functions can be used to calculate double integrals and triple integrals, which can be used to calculate the volume of a 3D object.
11.They can also be used to calculate line integrals, which can be used to calculate the amount of work done by a force.
12. Functions of several variables can also be used to calculate surface integrals, which can be used to calculate the amount of flux through a surface.
13. These functions can also be used to calculate vector fields, which can be used to describe the movement of particles in a fluid.
14. Functions of several variables can be used to solve systems of equations, such as those involving multiple variables.
15. These functions can also be used to solve optimization problems, such as finding the maximum or minimum of a function.

Young's theorem

Young's theorem, also known as Young's inequality, is a fundamental result in mathematical analysis, particularly in the study of integral inequalities and convex functions. There are different versions of Young's theorem, depending on the context in which it is applied. Here are two common forms:

Young's theorem, or Young's inequality, provides a useful relationship between products of numbers and their respective powers or, in the context of convolutions, between functions and their integrals.

1. Young's Inequality for Products

This version of the theorem is particularly useful in dealing with products of numbers raised to different powers. It states:

For any $a, b \geq 0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

Explanation:

- a and b are non-negative numbers.
- p and q are conjugate exponents, meaning their reciprocals add up to 1 ($\frac{1}{p} + \frac{1}{q} = 1$).
- The inequality states that the product of a and b is less than or equal to the sum of a raised to the power p divided by p and b raised to the power q divided by q.

This inequality helps in various applications where handling products and sums of powers is necessary, providing a bound that can simplify many calculations.

2. Young's Inequality for Convolutions

This form of the inequality is particularly useful in the analysis of functions, especially in the context of Lebesgue spaces and convolutions. It states:

Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with $1 \leq p, q, r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then the convolution $f * g \in L^r(\mathbb{R}^n)$, and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Explanation:

- f and g are functions in Lebesgue spaces L^p and L^q, respectively.
- The exponents p, q, r satisfy the relationship $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$.
- The convolution of f and g, denoted $f * g$, is a function that combines f and g in a specific way, commonly used in signal processing and differential equations.
- The inequality states that the L^r norm (a measure of the size) of the convolution $f * g$ is less than or equal to the product of the L^p norm of f and the L^q norm of g.

This version of Young's inequality is instrumental in proving various results in analysis, especially when dealing with convolutions and integrals in functional spaces.

Intuitive Understanding

At its core, Young's theorem helps in understanding and bounding the interactions between different mathematical quantities, whether they are simple numbers or more complex functions. It provides a way to control these interactions by relating them to more manageable expressions, making it a powerful tool in both theoretical and applied mathematics.

Implicit Function Theorem

The Implicit Function Theorem is a fundamental result in calculus, particularly in multivariable calculus, that provides conditions under which a relation defines a function implicitly. Here's an overview of the theorem and its implications:

Statement of the Implicit Function Theorem

Consider a function $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ defined by
$F(x, y) = 0,$
where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.

Let $(a, b) \in \mathbb{R}^{n+m}$ be a point such that $F(a, b) = 0$.

The Implicit Function Theorem states that if the $m \times m$ Jacobian matrix $\frac{\partial F}{\partial y}(a, b)$ is invertible (i.e., its determinant is non-zero), then there exist open sets $U \subset \mathbb{R}^n$ containing a and $V \subset \mathbb{R}^m$ containing b, and a unique continuously differentiable function $g : U \to V$ such that for every $x \in U$,
$F(x, g(x)) = 0.$

Interpretation and Applications

1. **Local Solution**: The theorem guarantees that locally around the point (a, b), the equation $F(x, y) = 0$ can be solved for y as a function of x, i.e., $y = g(x)$.

2. **Uniqueness**: The function g is unique and continuously differentiable.

3. **Invertibility Condition**: The condition that the Jacobian matrix $\frac{\partial F}{\partial y}$ is invertible ensures that the relationship between x and y can be expressed as a function.

Example

Consider the equation $F(x, y) = x^2 + y^2 - 1 = 0$. This represents a circle of radius 1.

- To apply the theorem, we compute the partial derivatives:

$$\frac{\partial F}{\partial x} = 2x, \quad \frac{\partial F}{\partial y} = 2y.$$

- At the point $(x_0, y_0) = (0, 1)$, the Jacobian matrix is $\frac{\partial F}{\partial y} = 2$, which is non-zero.

- Therefore, around $(0, 1)$, there exists a function $y = g(x)$ such that $x^2 + g(x)^2 = 1$.

Generalizations

The Implicit Function Theorem can be generalized to more complex settings, including higher dimensions and more abstract spaces, often encountered in advanced calculus and differential geometry.

Understanding and applying the Implicit Function Theorem is crucial in various fields such as physics, engineering, and economics, where relationships between variables are often expressed implicitly.

Properties of the Linearly Homogeneous Production Function

Let us suppose that a firm uses two inputs, labour (L) and capital (K), to produce its output (Q), and its production function is

$Q = f(L,K)$ (8.122)

[where L and K are quantities used of inputs labour (L) and capital (K) and Q is the quantity of output produced]

The function (8.122) is homogeneous of degree n if we have

$f(tL, tK) = tn\, f(L, K) = tnQ$ (8.123)

where t is a positive real number.

In the theory of production, the concept of homogenous production functions of degree one [n = 1 in (8.123)] is widely used. These functions are also called 'linearly' homogeneous production functions.

If the production function (8.122) is linearly homogeneous, then we would have

$f(tL, tK) = tf(L, K) = tQ$ (8.124)

From (8.124), it is clear that linear homogeneity means that raising of all inputs (independent variables) by the factor t will always raise the output (the value of the function) exactly by the factor t. Assumption of linear homogeneity, therefore, would amount to the assumption of constant returns to scale in economic theory.

Let us discuss below the properties of a linearly homogeneous production function as defined by (8.122) and (8.124).

Property I

The average (physical) product of labour (APL) and of capital (APK) can be expressed as functions of capital-labour ratio (K/L).

To prove this, let us multiply L and K in (8.122) by the factor t = 1/L. Then owing to linear homogeneity, we would have

$f(L/L, K/L) = Q/L$

$\Rightarrow f[(1, K/L)] = Q/L$

$\Rightarrow g(K/L) = APL \qquad (8.125)$

Similarly, we would have

$Q/K = h(L/K)$

$\Rightarrow APK = h(L/K) \qquad (8.126)$

Eqns. (8.125) and (8.126) give us that if L and K are increased by the firm in the same proportion, keeping the K/L ratio constant, then there would be absolutely no change in APL and APK, i.e., the APL and APK functions are homogeneous of degree zero in L and K.

Property II:

The marginal physical products, MPL and MPK, are also the functions of the K/L ratio.

We may establish this in the following way.

$Q = L.g(K/L)$ [from (8.125)] $\qquad (8.127)$

Therefore, we have

$MP_L = \partial Q/\partial L = g(K/L) + L.g'(K/L)(-K/L^2)$

$= g(K/L) - (K/L)g'(K/L)$

$= \Psi(K/L)$

Also, from (8.127), we have

$MPK = \partial Q/\partial K = Lg'(K/L)(1/L)$

$= g'(K/L)$

(8.128) and (8.129) establish Property II. That is, if the production function is homogeneous of degree one, then both MPL and MPK are homogeneous functions of K and L of degree zero.

Property III:

For the homogeneous production function of degree one as given by (8.122) and (8.124) we would have

We may establish this property in the following way. From (8.128) and (8.129), we have

$L(\partial Q/\partial L) + K(\partial Q/\partial K) = Q$

We may establish this property in the following way.

From (8.128) and (8.129), we have

$L(\partial Q/\partial L) + K(\partial Q/\partial K)$

$= Lg(K/L) - L(K/L)g'(K/L) + Kg'(K/L)$

$= Lg(K/L) - Kg'(K/L) + Kg'(K/L)$

$= Lg(K/L) = Q$ [from (8.127)]

(8.130) establishes property III, also known as the Euler's Theorem.

This property may also be stated like this. If the inputs labour (L) and capital (K) are paid at the rate of their respective marginal products ($\partial Q/\partial L$ and $\partial Q/\partial K$) then the total product (Q) would be exhausted, provided the production function is homogeneous of degree one.

Property IV:

The MRTS in the case of a homogeneous production function (of any degree n) is a function of K/L ratio, and the expansion path for such a function will be a straight line.

From (8.122) and (8.123), we have

$Q = f(L, K)$

and $f(tL, tK) = t^nQ$.

where t is a positive real number.

Since (8.122) has been assumed to be a homogeneous production function of degree n, we have got (8.123).

Now, putting $t = \frac{1}{L}$ in (8.123) we have

$$\frac{1}{L^n} Q = f\left(1, \frac{K}{L}\right) = g\left(\frac{K}{L}\right)$$

$$\text{or, } Q = L^n g\left(\frac{K}{L}\right) \tag{8.131}$$

From (8.131), we obtain:

$$\frac{\partial Q}{\partial L} = n\, L^{n-1} g\left(\frac{K}{L}\right) + L^n g'\left(\frac{K}{L}\right)\left(-\frac{K}{L^2}\right)$$

$$= n\, L^{n-1}\, g\left(\frac{K}{L}\right) - L^{n-2} K g'\left(\frac{K}{L}\right)$$

$$= L^{n-1}\left[ng\left(\frac{K}{L}\right) - \frac{K}{L} g'\left(\frac{K}{L}\right)\right] \tag{8.132}$$

Again, from (8.131), we have

$$\frac{\partial Q}{\partial K} = L^n g'\left(\frac{K}{L}\right)\left(\frac{1}{L}\right)$$

$$= L^{n-1}\, g'\left(\frac{K}{L}\right) \tag{8.133}$$

From (8.132) and (8.133), we have

$$MRTS_{L,K} = \frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = \frac{L^{n-1}\left[ng\left(\frac{K}{L}\right) - \frac{K}{L} g'\left(\frac{K}{L}\right)\right]}{L^{n-1}\, g'\left(\frac{K}{L}\right)}$$

$$= \frac{ng\left(\frac{K}{L}\right) - \frac{K}{L} g'\left(\frac{K}{L}\right)}{g'\left(\frac{K}{L}\right)} = \phi\left(\frac{K}{L}\right) \quad [\because n = \text{constant}] \tag{8.134}$$

Now, the equation of the expansion path is

$$\frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = \frac{r_L}{r_K} = \text{constant} \quad [\text{since } r_L \text{ and } r_K \text{ are the given input prices}] \tag{8.135}$$

$$\text{or } MRTS_{L,K} = \phi\left(\frac{K}{L}\right) = \text{constant} \qquad [\text{from (13) and (14)}] \tag{8.136}$$

That is, along the expansion path MRTSL,K = φ (K/L) = constant, i.e., along this path ratio is constant which implies that the expansion path is a straight line from the origin. Hence, Property IV is established. It may be noted that if we put n = 1 in above calculations, we would be able to establish the property when the production function is linearly homogeneous. Let us note here that the expansion path of the firm and the ridge lines are all isoclines by definition, and the equation of an isocline in general, is

$$\text{MRTS}_{L,K} = \frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = \text{constant} \qquad (8.135a)$$

Also, for a particular isocline, viz., the upper and the lower rigid line, the equations are respectively

$$\text{MRTS}_{L,K} = \frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = \frac{r_L}{r_K} = \text{constant} \qquad [\text{eq. } (8.135)]$$

And for the other two particular isoclines, viz., the upper and the lower ridge lines, the equations are respectively

$$\text{MRTS}_{L,K} = \frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = \infty = \text{constant} \qquad (8.135b)$$

$$\text{and } \text{MRTS}_{L,K} = \frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = 0 = \text{constant} \qquad (8.135c)$$

Therefore, the argument that we have put forward above to establish that the expansion path of a homogeneous production function (of any degree), is a straight line may be applied also to give us that under such a production function the isoclines, in general, and the ridge lines, in particular, are all straight lines from the origin.

Property V:

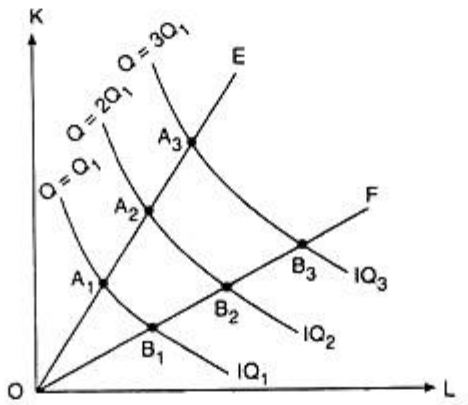
If the production function of the firm is linearly homogeneous, then the knowledge of the position of any one isoquant would enable us to obtain the whole IQ map of the firm. We may establish this property with the help of Figure.

In this figure, let us suppose that IQ1 is any isoquant of a firm for Q = Q1 and OE and OF are any two rays from the origin. The curve IQ1 has met these rays at the points A1 and B1 respectively. Let us suppose that the firm wants to have the IQ for Q = 2Q1. This can be achieved in the following way.

If the firm moves along the ray OE from the point A1 to A2 such that OA2 = 2. OA1, then at A2 both the input quantities would be doubled as compared to A1, and, therefore, if the production function is homogeneous of degree one, the firm's output quantity would also be doubled. That is, if at A1, the output is Q1, then at A2, the output would be 2Q1.

Similarly, if along the ray OF, we have, OB2 = 2. OB1 then the output of Q1 at B1 would be doubled and it would become 2Q1 at the point B2. Now the curve passing through the points B1, B2, etc. lying on the different rays would be the required IQ for Q = 2Q1.

In the same way, if the firm wants to have the IQ for Q = 3Qi, then along the rays like OE and OF it would have to move to points A3, B3, etc. such that OA3 = 3. OA1, OB3 = 3. OB1 and so on. As a result, the input quantities and the output quantity would also increase by the factor 3. Therefore, the curve passing through the points A3, B3, etc. would be the required IQ for Q = 3Q1.



The whole IQ map can be known from the knowledge of any one IQ

Therefore, that if the production function is linearly homogeneous, and the firm knows any one of its IQs for Q = Q1 (say), then it would be able to obtain the IQ for Q = tQ1 where t is a positive real number. Hence, Property V is established.

Euler's theorem

Euler’s theorem is a fundamental principle in economics, particularly in the theory of production and distribution. It is rooted in a mathematical formula attributed to Leonhard Euler, an 18th-century mathematician and physicist. Euler’s theorem is applied in various fields of economics, especially in understanding the distribution of income between the factors of production under specific conditions.

Definition of Euler’s Theorem

Euler’s theorem states that if a function $f(x_1, x_2, \ldots, x_n)$ is homogeneous of degree n, then the sum of all partial derivatives of the function, each multiplied by its corresponding variable, equals the original function multiplied by its degree of homogeneity. In terms of production function $F(K, L)$, which is homogeneous of degree one (constant returns to scale), Euler’s theorem implies that the income paid to each factor of production (capital, K, and labor, L), at their marginal products, exhausts the total product.

This mathematical representation provides a theoretical basis for understanding how income is distributed among the factors of production in an economy, assuming perfect competition and no externalities.

Example

Consider a simple economy where only two inputs, labor (L) and capital (K), are used to produce a certain good. The production function $F(K, L)$ is homogeneous of degree one, indicating constant returns to scale. According to Euler’s theorem, if the wage paid to labor equals its marginal product, MP_L, and the rent paid to capital equals its marginal product, MP_K,

This equation illustrates that the total output (or income) of the economy is exactly apportioned between labor and capital, according to their contribution to the production process.

Cobb – Douglas Production Function

The Cobb-Douglas production function is based on the empirical study of the American manufacturing industry made by Paul H. Douglas and C.W. Cobb. It is a linear homogeneous production function of degree one which takes into account two inputs, labour and capital, for the entire output of the .manufacturing industry.

The Cobb-Douglas production function is expressed as:

$Q = AL^{a}\ C^{\beta}$

where Q is output and L and C are inputs of labour and capital respectively. A, a and β are positive parameters where = a > O, β > O.

The equation tells that output depends directly on L and C, and that part of output which cannot be explained by L and C is explained by A which is the 'residual', often called technical change.

The production function solved by Cobb-Douglas had 1/4 contribution of capital to the increase in manufacturing industry and 3/4 of labour so that the C-D production function is $Q = AL^{3/4}\ C1^{/4}$

which shows constant returns to scale because the total of the values of L and C is equal to one: (3/4 + 1/4), i.e.,(a + β = 1) . The coefficient of labourer in the C-D function measures the percentage increase in (Q that would result from a 1 per cent increase in L, while holding C as constant.

Similarly, B is the percentage increase in Q that would result from a 1 per cent increase in C, while holding L as constant. The C-D production function showing constant returns to scale is depicted in Figure 20. Labour input is taken on the horizontal axis and capital on the vertical axis.

To produce 100 units of output, OC, units of capital and OL units of labour are used. If the output were to be doubled to 200, the inputs of labour and capital would have to be doubled. OC is exactly double of OC1 and of OL2 is double of OL2.

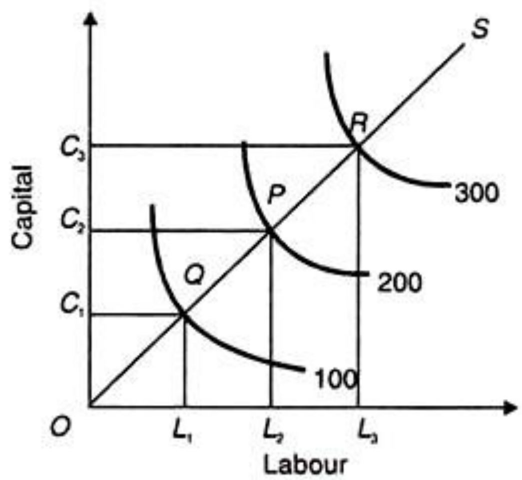
Similarly, if the output is to be raised three-fold to 300, the units of labour and capital will have to be increased three-fold. OC3 and OL3 are three times larger than OC1, and OL1, respectively. Another method is to take the scale line or expansion path connecting the equilibrium points Q, P and R. OS is the scale line or expansion path joining these points. It shows that the isoquants 100, 200 and 300 are equidistant. Thus, on the OS scale line OQ = QP = PR which shows that when capital and labour are increased in equal proportions, the output also increases in the same proportion.

Criticisms of C-D Production Function:

The C-D production function has been criticised by Arrow, Chenery, Minhas and Solow as discussed below:

1. The C-D production function considers only two inputs, labour and capital, and neglects some important inputs, like raw materials, which are used in production. It is, therefore, not possible to generalize this function to more than two inputs.

2. In the C-D production function, the problem of measurement of capital arises because it takes only the quantity of capital available for production. But the full use of the available capital can be made only in periods of full employment. This is unrealistic because no economy is always fully employed.



3. The C-D production function is criticised because it shows constant returns to scale. But constant returns to scale are not an actuality, for either increasing or decreasing returns to scale are applicable to production.

It is not possible to change all inputs to bring a proportionate change in the outputs of all the industries. Some inputs are scarce and cannot be increased in the same proportion as abundant inputs. On the other hand, inputs like machines, entrepreneurship, etc. are indivisible. As output increases due to the use of indivisible factors to their maximum capacity, per unit cost falls.

Thus when the supply of inputs is scarce and indivisibilities are present, constant returns to scale are not possible. Whenever the units of different inputs are increased in the production process, economies of scale and specialization lead to increasing returns to scale.

In practice, however, no entrepreneur will like to increase the various units of inputs in order to have a proportionate increase in output. His endeavour is to have more than proportionate increase in output, though diminishing returns to scale are also not ruled out.

4. The C-D production function is based on the assumption of substitutability of factors and neglects the complementarity of factors.

5. This function is based on the assumption of perfect competition in the factor market which is unrealistic. If, however, this assumption is dropped, the coefficients α and β do not represent factor shares.

6. One of the weaknesses of C-D function is the aggregation problem. This problem arises when this function is applied to every firm in an industry and to the entire industry. In this situation, there will be many production functions of low or high aggregation. Thus the C-D function does not measure what it aims at measuring.

Conclusion:

Thus the practicability of the C-D production function in the manufacturing industry is a doubtful proposition. This is not applicable to agriculture where for intensive cultivation, increasing the quantities of inputs will not raise output proportionately. Even then, it cannot be denied that constant returns to scale are a stage in the life of a firm, industry or economy. It is another thing that this stage may come after some time and for a short while.

It's Importance:

Despite these criticisms, the C-D function is of much importance.

1. It has been used widely in empirical studies of manufacturing industries and in inter-industry comparisons.

2. It is used to determine the relative shares of labour and capital in total output.

3. It is used to prove Euler's Theorem.

4. Its parameters a and b represent elasticity coefficients that are used for inter-sectoral comparisons.

5. This production function is linear homogeneous of degree one which shows constant returns to scale, If $\alpha + \beta = 1$, there are increasing returns to scale and if $\alpha + \beta < 1$, there are diminishing returns to scale.

6. Economists have extended this production function to more than two variables.

Constrained Optimization

Constrained optimization refers to the process of optimizing an objective function subject to constraints on the variables. This is a fundamental concept in fields like mathematics, economics, engineering, and operations research. Here's a brief overview of the key concepts and methods involved in constrained optimization:

Key Concepts

Objective Function: The function that needs to be maximized or minimized.

Constraints: These are the conditions that the solution must satisfy. They can be:

Equality Constraints: Conditions where a function of the variables equals a constant.

Inequality Constraints: Conditions where a function of the variables is either less than or greater than a constant.

Methods of Constrained Optimization

Lagrange Multipliers: This method involves converting a constrained problem into an unconstrained one by introducing additional variables called Lagrange multipliers. The solution is found by solving a system of equations derived from the original objective function and the constraints.

Kuhn-Tucker Conditions: These are necessary conditions for a solution in nonlinear programming to be optimal, provided certain regularity conditions are met. They generalize the method of Lagrange multipliers to handle inequality constraints.

Linear Programming: A method for optimizing a linear objective function, subject to linear equality and inequality constraints. The Simplex algorithm is a popular technique for solving linear programming problems.

Quadratic Programming: This involves optimizing a quadratic objective function subject to linear constraints. Techniques for solving quadratic programming problems include interior point methods and active set methods.

Gradient-Based Methods: These methods, such as Gradient Descent or Conjugate Gradient, are used for optimization problems where the objective function and constraints are differentiable.

Interior Point Methods: These methods are used for large-scale linear and nonlinear programming problems. They work by traversing the interior of the feasible region to find the optimal solution.

Penalty and Barrier Methods: These methods involve transforming a constrained optimization problem into a series of unconstrained problems by adding penalty terms to the objective function that penalize constraint violations.

Lagrangian Multiplier Technique

The Lagrangian Multiplier Technique is a mathematical method used in optimization problems to find the local maxima and minima of a function subject to equality constraints. It is particularly useful in situations where you need to optimize a function with one or more constraints.

Here's a step-by-step outline of the technique:

1. **Define the objective function and constraints:**
   - Let $f(x, y, \ldots)$ be the objective function to be maximized or minimized.
   - Let $g(x, y, \ldots) = 0$ be the constraint(s).
2. **Form the Lagrangian function:**
   - The Lagrangian function $\mathcal{L}$ is constructed by combining the objective function and the constraints using a new variable (the Lagrange multiplier, λ):

$$\mathcal{L}(x, y, \ldots, \lambda) = f(x, y, \ldots) + \lambda[g(x, y, \ldots) - c]$$

   where c is a constant value for the constraint function $g(x, y, \ldots)$.
3. **Take partial derivatives:**
   - Compute the partial derivatives of the Lagrangian function with respect to each variable and the Lagrange multiplier:

$$\frac{\partial \mathcal{L}}{\partial x}, \frac{\partial \mathcal{L}}{\partial y}, \ldots, \frac{\partial \mathcal{L}}{\partial \lambda}$$

4. **Set the partial derivatives to zero:**
   - Set each of these partial derivatives equal to zero to obtain a system of equations:

$$\frac{\partial \mathcal{L}}{\partial x} = 0, \frac{\partial \mathcal{L}}{\partial y} = 0, \ldots, \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

5. **Solve the system of equations:**
   - Solve this system of equations simultaneously to find the values of $x, y, \ldots,$ and λ.

6. Interpret the solutions:

   The solutions will give the points at which the objective function f is optimized subject to the constraint g.

Vector differentiation

The term vector differentiation essentially means the differentiation of a given vector function. Specifically in calculus, the derivatives of certain real functions are calculated in a real variable. If we consider a single variable function such as $y = g(z)$ then we can calculate y's derivative based on z. However, if we consider functions of several variables such as $y = g(z1, z2, \ldots, zn)$ then the derivative of y is calculated based on any of the provided independent variables. However, while doing this other independent variables should be taken as fixed.

Specifically, in the case of vector analysis, the derivatives of different functions are calculated based on any one of the independent variables. This is known as vector differentiation and it means that derivatives of a particular function of the following type are calculated.

$R(v) = x(v)\, i + y(v)\, j + z(v)\, k$, where R represents the position vector of a point $Q(x, y, z)$ in space that moves as v increases. It traces a particular curve in space. This can be corresponded with the parametric representation of a particular space curve which can be given as $x = x(v)$, $y = y(v)$, $z = z(v)$.

For finding derivatives we take the limit $\Delta R/\Delta v = [R(v + \Delta v) - R(v) / \Delta v$. If the limit of this function exists then the derivative is given by $dR/dv = (dx/dv)i + (dy/dv)j + (dz/dv)k$, where R is being differentiated on the basis of time (v denoting time).

Matrix Differentiation

Matrix differentiation is an extension of the concept of differentiation from calculus to functions of matrices. It is used in various fields such as optimization, statistics, and machine learning. Here are some key concepts and rules related to matrix differentiation:

Key Concepts

1. **Gradient**:
   - For a scalar function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$, the gradient is the matrix of partial derivatives of f with respect to the matrix X:

$$\nabla_X f = \frac{\partial f}{\partial X}$$

2. **Jacobian**:
   - For a vector-valued function $f : \mathbb{R}^{m\times n} \to \mathbb{R}^p$, the Jacobian matrix is the matrix of all first-order partial derivatives:

$$J_f = \frac{\partial \mathbf{f}}{\partial X}$$

3. **Hessian**:
   - For a scalar function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$, the Hessian is a second-order tensor (or matrix) of second-order partial derivatives:

$$H_f = \frac{\partial^2 f}{\partial X^2}$$

Basic Rules

1. **Scalar-Scalar**:
   - If f is a scalar function of a scalar variable x, then the derivative is:

$$\frac{\partial f}{\partial x}$$

2. **Scalar-Vector**:
   - If f is a scalar function of a vector $\mathbf{x} \in \mathbb{R}^n$, then the gradient is a vector:

$$\nabla_{\mathbf{x}} f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right]^T$$

3. **Scalar-Matrix**:
   - If f is a scalar function of a matrix $X \in \mathbb{R}^{m\times n}$, then the gradient is a matrix:

$$\nabla_X f = \left[\frac{\partial f}{\partial X_{ij}}\right]_{i=1,\ldots,m;j=1,\ldots,n}$$

4. **Product Rule**:

   - If $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, and $C = AB$, then:

$$\frac{\partial C}{\partial A} = B^T \quad \text{and} \quad \frac{\partial C}{\partial B} = A^T$$

5. **Chain Rule**:

   - For composite functions, the chain rule is used. If $Y = g(X)$ and $Z = f(Y)$, then:

$$\frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial Y}\frac{\partial Y}{\partial X}$$

Utility maximization

Utility maximization is a fundamental concept in economics and decision theory. It refers to the idea that individuals and organizations seek to maximize their utility, which is a measure of the satisfaction or benefit derived from consuming goods and services, subject to their constraints such as income or resources.

Key Points of Utility Maximization

1. **Utility Function**: This represents the relationship between the quantity of goods consumed and the level of satisfaction or utility derived from them. It can be expressed mathematically as $U = f(x_1, x_2, ..., x_n)$, where U is the utility and $x_1, x_2, ..., x_n$ are the quantities of different goods.

2. **Constraints**: These are the limitations that individuals face, such as budget constraints. For example, if a person has a limited income, they must choose a combination of goods that maximizes their utility without exceeding their budget.

3. **Marginal Utility**: This is the additional satisfaction or utility gained from consuming one more unit of a good or service. According to the law of diminishing marginal utility, the marginal utility of consuming additional units of a good decreases as consumption increases.

4. **Optimization**: Utility maximization involves choosing the combination of goods that provides the highest utility. This can be done using calculus (Lagrange multipliers) or graphical methods (indifference curves and budget lines).

5. **Indifference Curves**: These curves represent combinations of goods that provide the same level of utility to the consumer. The consumer's goal is to reach the highest possible indifference curve given their budget constraint.

6. **Budget Line**: This line represents all the combinations of goods that a consumer can afford given their income and the prices of goods. The optimal consumption point is where the budget line is tangent to the highest attainable indifference curve.

Profit maximization

Profit maximization is the process by which a company determines the price and production level that returns the greatest profit. This concept is a key goal in traditional economic theories, especially within the realm of microeconomics. Here are some key points:

Key Concepts

1. **Total Revenue (TR) and Total Cost (TC):**

   - Total Revenue is the total income a company generates from selling its products or services (Price x Quantity).
   - Total Cost includes all the expenses incurred in the production process (fixed and variable costs).

2. **Profit:**

   - Profit is the difference between Total Revenue (TR) and Total Cost (TC).
   - Profit=Total Revenue−Total Cost\ {Profit} = {Total Revenue} - {Total Cost}Profit=Total Revenue−Total Cost

3. **Marginal Revenue (MR) and Marginal Cost (MC):**

   - Marginal Revenue is the additional revenue gained from selling one more unit of a product.
   - Marginal Cost is the additional cost incurred from producing one more unit of a product.
   - The profit-maximizing output level is where Marginal Revenue (MR) equals Marginal Cost (MC): MR=MC

Cost minimization

Cost minimization is a strategy or approach used by businesses and organizations to reduce expenses and optimize resources while maintaining productivity and quality. It involves identifying areas where costs can be cut or minimized without compromising the efficiency or effectiveness of operations. This can include measures such as:

1. **Streamlining Processes:** Identifying and eliminating unnecessary steps or inefficiencies in workflows and operations.
2. **Negotiating with Suppliers:** Negotiating better deals with suppliers for raw materials, equipment, or services to lower procurement costs.
3. **Optimizing Inventory:** Managing inventory levels efficiently to reduce holding costs while ensuring adequate stock to meet demand.
4. **Utilizing Technology:** Implementing technology solutions such as automation, digital tools, and software systems to improve efficiency and reduce manual labor costs.
5. **Energy Efficiency:** Implementing energy-saving measures to reduce utility costs, such as using energy-efficient equipment and optimizing lighting and heating/cooling systems.
6. **Outsourcing and Offshoring:** Exploring outsourcing or offshoring options for non-core functions to lower labor costs while maintaining quality standards.
7. **Training and Development:** Investing in training programs to enhance employee skills and productivity, reducing errors and rework costs.
8. **Monitoring and Analysis:** Regularly monitoring expenses, analyzing cost drivers, and identifying opportunities for further cost reduction.
9. **Benchmarking:** Comparing costs and performance metrics with industry peers or best practices to identify areas for improvement.

Cost minimization is a continuous process that requires ongoing evaluation, adaptation, and decision-making to achieve sustainable cost reductions while supporting organizational goals and objectives.

UNIT IV

LINEAR AND NON-LINEAR PROGRAMMING

Optimization with Inequality Constraints

Optimization with inequality constraints involves finding the maximum or minimum of an objective function subject to certain restrictions, which are represented as inequalities. This type of optimization is common in various fields such as economics, engineering, and operations research. Here's a brief overview of the key concepts:

1. **Problem Formulation**

An optimization problem with inequality constraints can be formulated as:

$\min_x f(x)$
subject to:
$g_i(x) \leq 0, \quad i = 1, 2, \ldots, m$
where:

- $f(x)$ is the objective function to be minimized.
- $g_i(x)$ are the inequality constraint functions.

2. **Feasible Region**

The feasible region is the set of all points x that satisfy the inequality constraints. The optimization is only performed over this region.

3. **Karush-Kuhn-Tucker (KKT) Conditions**

The KKT conditions are necessary conditions for a solution to be optimal, provided that some regularity conditions are met. For a problem with inequality constraints, the KKT conditions include:

- **Primal feasibility**: $g_i(x) \leq 0$
- **Dual feasibility**: $\lambda_i \geq 0$
- **Complementary slackness**: $\lambda_i g_i(x) = 0$
- **Stationarity**: $\nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0$

Here, λ_i are the Lagrange multipliers associated with the inequality constraints.

4. **Lagrangian Function**

The Lagrangian function incorporates the objective function and the constraints:

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

5. **Solving the Problem**

There are various methods to solve optimization problems with inequality constraints:

- **Interior Point Methods**: These methods approach the solution from within the feasible region and are effective for large-scale problems.
- **Barrier Methods**: These methods add a barrier term to the objective function that penalizes solutions close to the boundary of the feasible region.
- **Sequential Quadratic Programming (SQP)**: This method solves a sequence of quadratic approximations to the original problem.

6. **Examples**

Example 1: Quadratic Programming Problem
Minimize: $f(x) = x_1^2 + x_2^2$
Subject to: $x_1 + x_2 \geq 1$ and $x_1, x_2 \geq 0$

Example 2: Linear Programming Problem
Minimize: $f(x) = c^T x$
Subject to: $Ax \leq b$

Applications

Optimization with inequality constraints is widely used in various fields, including:

Engineering Design: To optimize the design parameters while satisfying safety and performance constraints.

Economics: To maximize utility or profit subject to budget constraints.

Operations Research: To optimize resource allocation and scheduling while satisfying operational constraints.

Understanding these principles and methods is crucial for effectively solving optimization problems with inequality constraints.

1 Inequality Constraints

1.1 One Inequality constraint

Problem: maximize $f(x, y)$ subject to $g(x, y) \leq b$.

As we see here the constraint is written as inequality instead of equality.

An inequality constraint $g(x, y) \leq b$ is called *binding (or active) at a point* (x, y) if $g(x, y) = b$ and *not binding* (or inactive) if $g(x, y) < b$.

Again we consider the same Lagrangian function

$$L(x, y, \lambda) = f(x, y) - \lambda[g(x, y) - b].$$

Theorem 1 *Suppose (x^*, y^*) is a solution of the above problem: (x^*, y^*) maximizes f on the constraint set $g^*(x, y) \leq b$.*

*Suppose the following **qualification** is satisfied: If $g(x^*, y^*) = b$ (i.e. if (x^*, y^*) is binding) then $Dg(x^*, y^*) \neq (0, 0)$. Then there exists a multiplier λ^* such that*

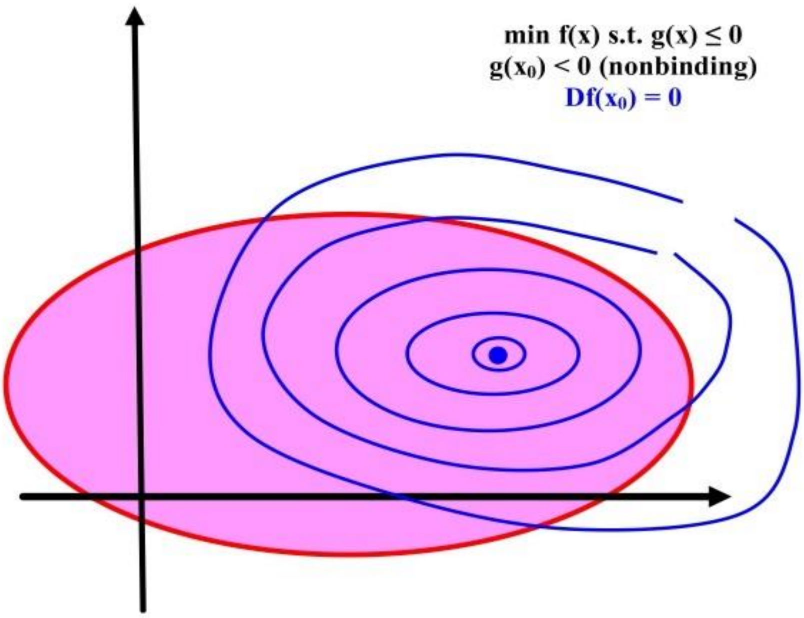
$$\begin{array}{ll} (a) & \frac{\partial L}{\partial x}(x^*, y^*, \lambda^*)) = 0, \\ (b) & \frac{\partial L}{\partial y}(x^*, y^*, \lambda^*)) = 0, \\ (c) & \lambda^*[g(x^*, y^*) - b] = 0, \\ (d) & \lambda^* \geq 0, \\ (e) & g(x^*, y^*) \leq b. \end{array}$$

Remark 1. These conditions, as well as the conditions from theorems bellow concerning with inequality conditions are called Karush-Kuhn-Tucker (KKT) conditions.

Remark 2. For the minimization problem the condition (d) must be replaced by
(d') $\lambda^* \leq 0$.

Almost a proof. Consider the following two cases: (x^*, y^*) is binding or not binding.

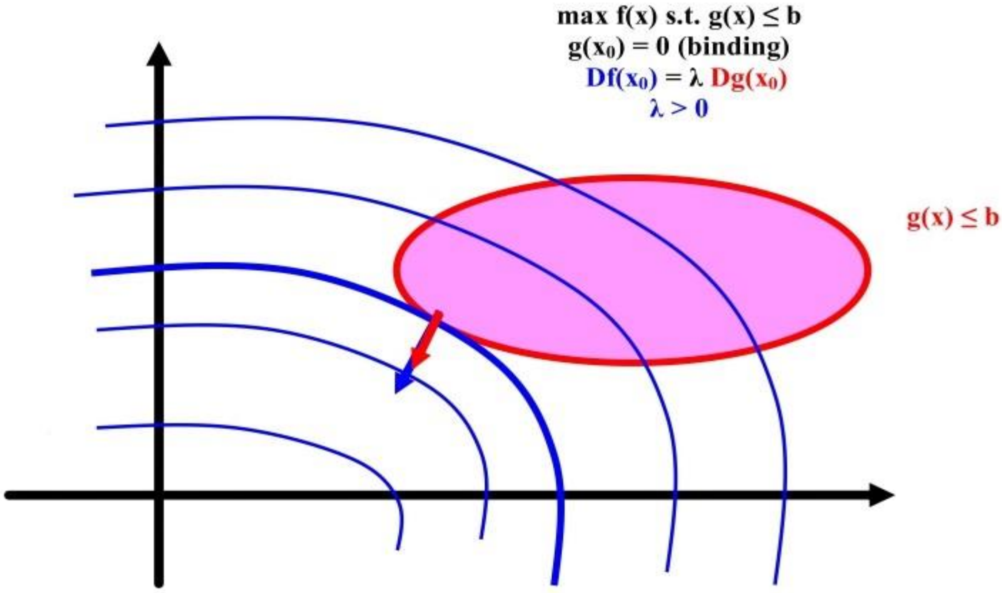
Case 1: (x^*, y^*) is not binding $g(x^*, y^*) < 0$.



This means that (x^*, y^*) is an *inner* (unconstraint) maximum, thus $f_x(x^*, y^*) = 0,\ f_y(x^*, y^*) = 0$. In this case we *can take* $\lambda = 0$ the conditions (a) - (e) are satisfied. Indeed

(a) $L_x(x^*, y^*, \lambda^*)) = f_x(x^*, y^*) - \lambda \cdot g_x(x^*, y^*) = f_x(x^*, y^*) - 0 \cdot g_x(x^*, y^*) = 0;$
(b) $L_y(x^*, y^*, \lambda^*)) = f_y(x^*, y^*) - \lambda \cdot g_y(x^*, y^*) = f_y(x^*, y^*) - 0 \cdot g_x(x^*, y^*) = 0;$
(c)$\lambda^* \cdot [g(x^*, y^*) - b] = 0 \cdot [g(x^*, y^*) - b] = 0;$
(d) $0 = \lambda^* \geq 0;$
(e) $g(x^*, y^*) - b < 0.$

Case 2: (x^*, y^*) is binding $g(x^*, y^*) = 0$.



This means that (x^*, y^*) is a maximizer constrained by the equality condition, thus there exists λ^* such that

$$L_x(x^*, y^*) = 0,\ L_y(x^*, y^*) = 0,\ L_\lambda(x^*, y^*) = 0,$$

this again implies the needed conditions (a) - (e):

(a) $L_x(x^*, y^*, \lambda^*)) = 0$;
(b) $L_y(x^*, y^*, \lambda^*)) = 0$;
(c)$\lambda^* \cdot [g(x^*, y^*) - b] = \lambda^* \cdot 0 = 0$;
(d) since of maximality of (x^*, y^*) the gradients $\nabla f(x^*, y^*)$ and $\nabla g(x^*, y^*)$ must have the same directions, thus $\lambda \geq 0$;
(e) $g(x^*, y^*) - b = 0$.

Remark. What is the meaning of the zero $\lambda = 0$ multiplier in Case 1? The shadow price in this case is 0: the maximal value $f(x^*, y^*)$ does not change when we change b a little.

Example 1. Minimize $f(x, y) = x^2 + y^2$ subject of $g(x, y) = 2x + y \leq 2$.
Solution. There are no critical points of g at all, so the qualification is satisfied.

The lagrangian in this case is

$$L(x, y, \lambda) = x^2 + y^2 - \lambda(2x + y - 2),$$

and the KKT conditions from Theorem are

$$\begin{array}{ll} (a) & \frac{\partial L}{\partial x}(x, y, \lambda) = 2x - 2\lambda = 0, \\ (b) & \frac{\partial L}{\partial y}(x, y, \lambda) = 2y - \lambda y = 0, \\ (c) & \lambda[g(x, y) - b] = \lambda(2x + y - 2) = 0, \\ (d) & \lambda \leq 0, \\ (e) & g(x, y) = 2x + y \leq 2. \end{array}$$

We consider two cases:

Case 1. $\lambda = 0$, in this case our system looks as

$$\begin{array}{ll} (a) & x = 0, \\ (b) & y = 0, \\ (c) & 0 = 0, \\ (d) & 0 \leq 0, \\ (e) & 2x + y \leq 2, \end{array}$$

so the solution in this case is $(x, y, \lambda) = (0, 0, 0)$.
Case 2. $2x + y - 2 = 0$, in this case our system looks as

$$\begin{array}{ll} (a) & 2x = 2\lambda, \\ (b) & 2y = \lambda, \\ (c) & 2x + y - 2 = 0, \\ (d) & \lambda \leq 0, \\ (e) & 2x + y \leq 2, \end{array}$$

so $x = 2y,\ 2x+y = 2$, this gives the solution $x = 0.8,\ y = 0.4$ but $\lambda = 0.8 > 0$ so this solution *can not* be a minimizer.

So if this constrained minimization problem has a solution, it *can be* only $(0,0)$.

1.2 Two Inequality Constraints

Maybe it will be useful to consider separately the following problem:

$$max\ f(x_1,x_2,x_3)=0\ \ s.t.\ \ g_1(x_1,x_2,x_3)\le b_1,\ \ g_2(x_1,x_2,x_3)\le b_2.$$

Lagrangian function in this case is

$$\begin{array}{l} L(x_1,x_2,x_3)= \\ f(x_1,x_2,x_3)-\lambda_1(g_1(x_1,x_2,x_3)-a)-\lambda_2(g_2(x_1,x_2,x_3)-b_2). \end{array}$$

The KKT conditions in this case look as

$$\begin{array}{ll}
(1) & \frac{\partial}{\partial x_1}f(x_1,x_2,x_3)-\lambda_1\frac{\partial}{\partial x_1}g_1(x_1,x_2,x_3)-\lambda_2\frac{\partial}{\partial x_1}g_2(x_1,x_2,x_3)=0 \\
(2) & \frac{\partial}{\partial x_2}f(x_1,x_2,x_3)-\lambda_1\frac{\partial}{\partial x_2}g_1(x_1,x_2,x_3)-\lambda_2\frac{\partial}{\partial x_2}g_2(x_1,x_2,x_3)=0 \\
(3) & \frac{\partial}{\partial x_3}f(x_1,x_2,x_3)-\lambda_1\frac{\partial}{\partial x_3}g_1(x_1,x_2,x_3)-\lambda_2\frac{\partial}{\partial x_3}g_2(x_1,x_2,x_3)=0 \\
(4) & \lambda_1[g_1(x_1,x_2,x_3)-b_1]=0 \\
(5) & \lambda_2[g_2(x_1,x_2,x_3)-b_2]=0 \\
(6) & \lambda_1\ge 0 \\
(7) & \lambda_2\ge 0 \\
(8) & g_1(x_1,x_2,x_3)\le b_1 \\
(9) & g_2(x_1,x_2,x_3)\le b_2.
\end{array}$$

Consider, concerning complementary slackness conditions (4) and (5), the following 4 cases:
Case 1: $\lambda_1=0,\ \lambda_2=0.$
Case 2: $g_1(x_1,x_2,x_3)-b_1=0,\ \lambda_2=0.$
Case 3: $\lambda_1=0,\ g_2(x_1,x_2,x_3)-b_2=0.$
Case 4: $g_1(x_1,x_2,x_3)-b_1=0,\ g_2(x_1,x_2,x_3)-b_2=0.$

We rewrite the KKT conditions in these cases:

Case 1: $\lambda_1 = 0,\ \lambda_2 = 0.$

$$\begin{array}{ll}
(1) & \frac{\partial}{\partial x_1} f(x_1, x_2, x_3) = 0 \\
(2) & \frac{\partial}{\partial x_2} f(x_1, x_2, x_3) = 0 \\
(3) & \frac{\partial}{\partial x_3} f(x_1, x_2, x_3) = 0 \\
(4) & \\
(5) & \\
(6) & \\
(7) & \\
(8) & g_1(x_1, x_2, x_3) \le b_1 \\
(9) & g_2(x_1, x_2, x_3) \le b_2
\end{array}$$

so in this case we face ordinary nonconstrained optimization problem

$$max \ \ f(x_1, x_2, x_3),$$

but with additional conditions $g_1(x_1, x_2, x_3) \le b_1$ and $g_2(x_1, x_2, x_3) \le b_2$, that is ignore all candidates (critical points of f) which are out of feasible region.

Case 2: $g_1(x_1, x_2, x_3) - b_1 = 0,\ \lambda_2 = 0.$

$$\begin{array}{ll}
(1) & \frac{\partial}{\partial x_1} f(x_1, x_2, x_3) - \lambda_1 \frac{\partial}{\partial x_1} g_1(x_1, x_2, x_3) = 0 \\
(2) & \frac{\partial}{\partial x_2} f(x_1, x_2, x_3) - \lambda_1 \frac{\partial}{\partial x_2} g_1(x_1, x_2, x_3) = 0 \\
(3) & \frac{\partial}{\partial x_3} f(x_1, x_2, x_3) - \lambda_1 \frac{\partial}{\partial x_3} g_1(x_1, x_2, x_3) = 0 \\
(4) & g_1(x_1, x_2, x_3) - b_1 = 0 \\
(5) & \\
(6) & \lambda_1 \ge 0 \\
(7) & \\
(8) & \\
(9) & g_2(x_1, x_2, x_3) \le b_2
\end{array}$$

so in this case we face the problem with one equality constraint

$$max \ \ f(x_1, x_2, x_3) \ \ s.t. \ \ g_1(x_1, x_2, x_3) = b_1$$

with additional conditions $g_2(x_1, x_2, x_3) \le b_2, \ \ \lambda_1 \ge 0$, that is we ignore all candidates with $g_2(x_1, x_2, x_3) > b_2$ or $\lambda_1 < 0$.

Case 3: $\lambda_1 = 0,\ g_2(x_1, x_2, x_3) - b_2 = 0.$

$$\begin{array}{ll}
(1) & \frac{\partial}{\partial x_1} f(x_1, x_2, x_3) - \lambda_1 \frac{\partial}{\partial x_1} g_1(x_1, x_2, x_3) = 0 \\
(2) & \frac{\partial}{\partial x_2} f(x_1, x_2, x_3) - \lambda_1 \frac{\partial}{\partial x_2} g_1(x_1, x_2, x_3) = 0 \\
(3) & \frac{\partial}{\partial x_3} f(x_1, x_2, x_3) - \lambda_1 \frac{\partial}{\partial x_3} g_1(x_1, x_2, x_3) = 0 \\
(4) & \\
(5) & g_2(x_1, x_2, x_3) - b_2 = 0 \\
(6) & \\
(7) & \lambda_2 \ge 0 \\
(8) & g_1(x_1, x_2, x_3) \le b_1 \\
(9) &
\end{array}$$

so in this case we face the problem with one equality constraint

$$max \quad f(x_1, x_2, x_3) \quad s.t. \quad g_2(x_1, x_2, x_3) = b_2$$

with additional conditions $g_1(x_1, x_2, x_3) \leq b_1, \quad \lambda_2 \geq 0$, that is we ignore all candidates with $g_1(x_1, x_2, x_3) > b_2$ or $\lambda_2 < 0$.

Case 4: $g_1(x_1, x_2, x_3) - b_1 = 0, \; g_2(x_1, x_2, x_3) - b_2 = 0$.

$$\begin{array}{ll}
(1) & \frac{\partial}{\partial x_1} f(x_1, x_2, x_3) - \lambda_1 \frac{\partial}{\partial x_1} g_1(x_1, x_2, x_3) - \lambda_2 \frac{\partial}{\partial x_1} g_2(x_1, x_2, x_3) = 0 \\
(2) & \frac{\partial}{\partial x_2} f(x_1, x_2, x_3) - \lambda_1 \frac{\partial}{\partial x_2} g_1(x_1, x_2, x_3) - \lambda_2 \frac{\partial}{\partial x_2} g_2(x_1, x_2, x_3) = 0 \\
(3) & \frac{\partial}{\partial x_3} f(x_1, x_2, x_3) - \lambda_1 \frac{\partial}{\partial x_3} g_1(x_1, x_2, x_3) - \lambda_2 \frac{\partial}{\partial x_3} g_2(x_1, x_2, x_3) = 0 \\
(4) & g_1(x_1, x_2, x_3) - b_1 = 0 \\
(5) & g_2(x_1, x_2, x_3) - b_2 = 0 \\
(6) & \lambda_1 \geq 0 \\
(7) & \lambda_2 \geq 0 \\
(8) & \\
(9) &
\end{array}$$

so in this case we face the problem with two equality constraint

$$max \quad f(x_1, x_2, x_3) \quad s.t. \quad g_1(x_1, x_2, x_3) = b_1, \quad g_2(x_1, x_2, x_3) = b_2$$

with additional conditions $\lambda_1 \geq 0, \quad \lambda_2 \geq 0$, that is we ignore all candidates with $\lambda_1 < 0$ or $\lambda_2 < 0$.

Linear Programming

Linear Programming is a mathematical technique used for allocating limited resources in an optimum manner. In other words Linear Programming is a planning technique that permits some objective function to be minimized or maximized within the framework of given situational restrictions. Linear Programming is an important technique developed for optimum utilization of resources.

Linear Programming is a method for determining optimum values of a linear function subject to constraints expressed as linear equations or inequalities. Linear Programming technique was formulated by a Russian mathematician L.V. Kantorovich, but the present version of the simplex method was developed by Geoge B. Dentzig in 1947.

In military operation, the effect is to inflict maximum damage to the enemy at a minimum cost and loss. In an industry, the management always tries to utilize its resources in the best possible manner. A salaried person tries to make investments in such a manner that the returns on investment are high but at the same time income tax liability is kept low.

In all the above cases, if the constraints are represented by linear equations/inequations(in one, two or more variables), and a particular plan of action from several alternatives is to be chosen, we use Linear Programming. The word linear means that all inequalities using the function to be maximized or minimized are linear, and the word programming refers to planning(choosing amongst alternatives) rather than the computer programming sense.

The basic components of Linear Programming are mentioned below.

Decision Variables,

Constraints,

Data,

Objective Functions,

Feasible Region and Feasible Solution, and

Optimal Solution.

Properties of Linear Programming

The properties of Linear Programming are listed below.

Decision variable: They are represented by x, y, etc., and are referred to as limited resources.

Objective function: It is the linear function between decision variables, that is to be maximized or minimized. In most businesses the objective is to maximize profits or minimize cost.

Constrains: These are the restrictions or limits which are determined to make the situation optimum.

Non negativity: The relationship between variables must be linear.

Additivity: It is the total of all activities is the sum of each individual activity. None of the activities interact with one another.

Divisibility: Decision variables may have fractional values.

Deterministic or Certainty: All the parameters are assumed to be known exactly.

Finiteness: There are always a finite number of decision variables and constraints.

Linear Programming Methods

There are two methods to solve a Linear Programming Problem(LPP). These are explained below.

Simplex Method

The simplex method is used to obtain the optimal solution of a linear system of constraints, in a linear objective function. It works by beginning at the basic vertex of the feasible region, and then iteratively moving towards the adjacent vertices, also improving upon the solution each time until the optimal solution is found.

For example, The advertising alternatives for a company includes television, newspaper and radio advertisement. The cost for each medium with its audience coverage is provided below.

–	Television	Newspaper	Radio
Cost per advertisement ($)	2000	600	300
Audience per advertisement	100,000	40,000	18,000

The local newspaper limits the number of advertisements from a single company upto ten. Moreover, to balance the advertising among the three types of media, no more than the half of the total number of advertisements can occur on the radio. And at least of 10% should occur on televisions. The weekly advertising budget provided is $18,200. How many advertisements should be run in each of the three types of media so that the total audience is maximum?

Step 1: Identification of Decision Variables

Let X1, X2, X3

represent the total number of ads for television, newspaper, and radio respectively.

Step 2: Identification of Objective Function

The objective of the company is to maximize the audience.So the objective function is given as, $Z=100000X_1+40000X_2+18000X_3$

Step 3: Constraints

It is clear from the question that we have a budget constraint. The total provided budget is \$18,200, and the individual cost per advertisement for television, newspaper and radio advertisements is \$2000, \$600 and \$300 respectively.

This can be represented by the equation, $2000X_1+600X_2+300X_3\leq18200$

Now for a newspaper advertisement there is an upper cap on the number of advertisements to 10. So the first constraint is $X2<10$

The second constraint is the number of advertisements on television. As the company wants at least 10% of the total advertisements to be on television, so it can be represented as $X1\geq0.1(X1+X2+X3)$

The last constraint is the number of advertisements on the radio which cannot be more than half of the total number of advertisements. So it can be represented as $X3\leq0.5(X1+X2+X3)$

Now the Linear Programming Problem is formulated. We have reiterated all the constraints as follows.

$2000X_1+600X_2+300X_3\leq182002000X1+600X2+300X3\leq18200$

$X_2<10X2<10$

$X_1\geq0.1(X_1+X_2+X_3)\Rightarrow-9X_1+X_2+X_3\leq0X1\geq0.1(X1+X2+X3)\Rightarrow-9X_1+X_2+X_3\leq0$

$X_3\leq0.5(X_1+X_2+X_3)\Rightarrow-X_1-X_2+X_3\leq0$

As we have a total of 4 equations so to balance out each equation we are introducing 4 slack variables S1, S2, S3, and S4

Now our equations are as follows.

$2000X_1+600X_2+300X_3+S_1=18200$

$X_2+S_2=10$

$-9X_1+X_2+X_3+S_3=0$

$-X_1-X_2+X_3+S_4=0$

Now after solving these equations we get the values $X_1=4$, $X_2=10$, $X_3=14$

On solving the objective function we get the maximum weekly audience as 1,052,000, which is the required result.

Graphical Method

The graphical method involves formulating a set of linear inequalities subject to the constraints. Then the inequalities are plotted on an XY plane. Once we have plotted all the inequalities on a graph the intersection region gives us a feasible region. This feasible region explains what all values our model can take which also gives us the optimal solution.

For example, A farmer has recently acquired a piece of land of 110 hectares . He has decided to grow wheat and barley on this land. Due to the good quality of the sunlight and the region's excellent climate, the entire production of wheat and barley can be sold in the market. He wants to know how to plant each variety of the two crops in these 110 hectares, given the costs, net profits and labor requirements are provided in data shown below.

Variety	**Cost (Price/Hec)**	**Net Profit (Price/Hec)**	**Man-days/Hec**
Wheat	100	50	10
Barley	200	120	30

The farmer has a budget of \$10,000 and availability of 1,200 days during the planning horizon. What will be the optimal solution and the optimal value?

Step 1: Identification of Decision Variables

X (in hectares) = Total area for growing wheat

Y (in hectares) = Total area for growing barley

So X and Y are the decision variables.

Step 2: Identification of Objective Function

As the production from the entire land can be sold in the market. The farmer would want to maximize the profit of his total produce. We are given net profit for both the crops. The farmer earns a net profit of \$50 for each hectare of wheat and \$120 for each hectare of barley.

Thus, our objective function is, Max $Z = 50X + 120Y$

Step 3: Constraints

We have an upper cap on the total cost spent by the farmer due to the given budget.

So we can write the equations as, $100X+200Y\leq 10{,}000$

The next constraint is the upper cap on the availability of the total number of days for the planning horizon. As the total number of days available is 1200, so the equation will be, $10X+30Y\leq 1200$

The third constraint is the total area given for plantation. The total available area is 110 hectares and thus the equation becomes, $X+Y\leq 110$

Step 4: Non-negativity Restriction

The values of X and Y must be greater than or equal to 0, so $X\geq 0$, $Y\geq 0$

Now we solve this formulated LPP.

$100X+200Y \leq 10{,}000 \Rightarrow X+2Y \leq 100$

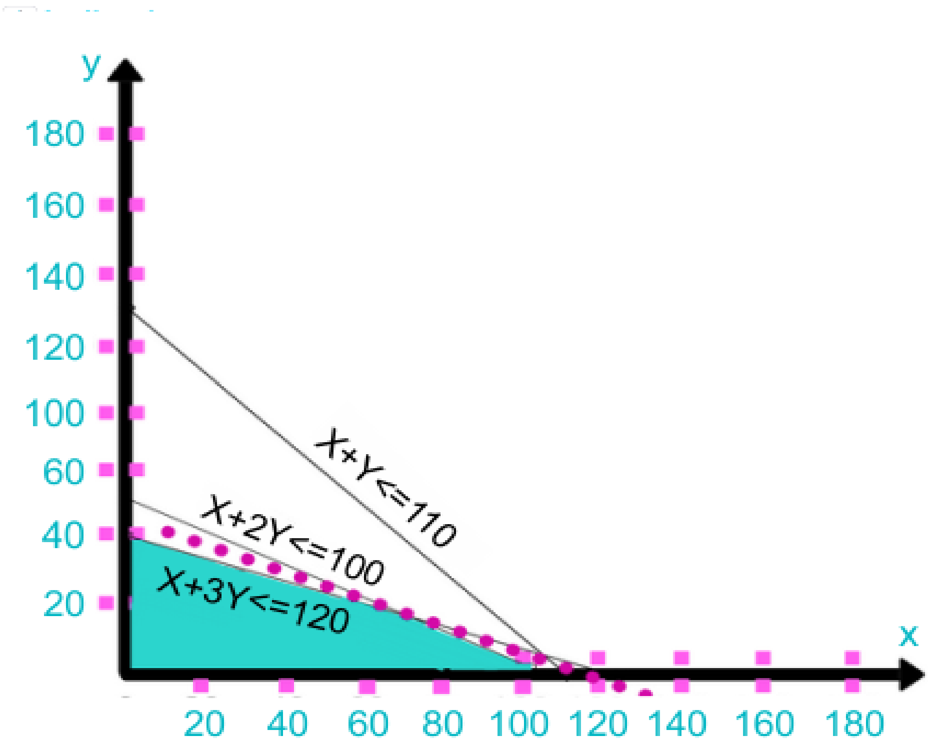
$10X+30Y \leq 1200 \Longrightarrow X+3Y \leq 120$

The third equation is already in its simplified form, $X+Y \leq 110$

The first 2 lines on a graph are plotted in the first quadrant. The optimal feasible solution is achieved at the point of intersection where the budget & number of days constraints are active. This means the point at which the equations $X+2Y \leq 100$$X+2Y \leq 100$ and $X+3Y \leq 120$ intersect gives us the optimal solution.

The values for X and Y which give the optimal solution is at (60,20). Thus to maximize profit the farmer should produce wheat and barley in 60 hectares and 20 hectares of land respectively.

The maximum profit is, Max Z = $50 \times 60+120 \times 20=\$5400$



In the above graph, the blue colored part is the feasible region which indicates the maximum profit.

Uses of Linear Programming

Linear Programming has vast applications on industry level as listed below.

1. It is used to solve questions on matching diets to nutritional and other additional constraints with a minimum amount of changes.
2. This mathematical technique allows the generation of optimal solutions that satisfy several constraints at once.
3. Linear Programming is used in the Food and agriculture industry where farmers can generate more revenues for their land for making profits.
4. The transportation industry works on linear programming techniques for cost and time efficiency.
5. Manufacturing units work on this technique to generate more profit for the company.

Simplex Method

Introduction

The Simplex method is an approach to solving linear programming models by hand using slack variables, tableaus, and pivot variables as a means to finding the optimal solution of an optimization problem. A linear program is a method of achieving the best outcome given a maximum or minimum equation with linear constraints. Most linear programs can be solved using an online solver such as MatLab, but the Simplex method is a technique for solving linear programs by hand. To solve a linear programming model using the Simplex method the following steps are necessary:

- Standard form
- Introducing slack variables
- Creating the tableau
- Pivot variables
- Creating a new tableau
- Checking for optimality
- Identify optimal values

This document breaks down the Simplex method into the above steps and follows the example linear programming model shown below throughout the entire

$$
\begin{aligned}
Minimize: -z &= -8x_1 - 10x_2 - 7x_3 \\
s.t.: x_1 + 3x_2 + 2x_3 &\leq 10 \\
-x_1 - 5x_2 - x_3 &\geq -8 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

document to find the optimal solution.

Step 1: Standard Form

Standard form is the baseline format for all linear programs before solving for the optimal solution and has three requirements: (1) must be a maximization problem, (2) all linear constraints must be in a less-than-or-equal-to inequality, (3) all variables are non-negative. These requirements can always be satisfied by transforming any given linear program using basic algebra and substitution. Standard form is necessary because it creates an ideal starting point for solving the Simplex method as efficiently as possible as well as other methods of solving optimization problems.

To transform a minimization linear program model into a maximization linear program model, simply multiply both the left and the right sides of the objective function by -1.

$$
\begin{gathered}
-1 \times (-z = -8x_1 - 10x_2 - 7x_3) \\
z = 8x_1 + 10x_2 + 7x_3 \\
Maximize: z = 8x_1 + 10x_2 + 7x_3
\end{gathered}
$$

Transforming linear constraints from a greater-than-or-equal-to inequality to a less-than-or-equal-to inequality can be done similarly as what was done to the objective function. By multiplying by -1 on both sides, the inequality can be changed to less-than-or-equal-to.

$$
\begin{gathered}
-1 \times (-x_1 - 5x_2 - x_3 \geq -8) \\
x_1 + 5x_2 + x_3 \leq 8
\end{gathered}
$$

Once the model is in standard form, the slack variables can be added as shown in Step 2 of the Simplex method.

Step 2: Determine Slack Variables

Slack variables are additional variables that are introduced into the linear constraints of a linear program to transform them from inequality constraints to equality constraints. If the model is in standard form, the slack variables will always have a +1 coefficient. Slack

$$x_1 + 3x_2 + 2x_3 + \mathbf{s_1} = 10$$
$$x_1 + 5x_2 + x_3 + \mathbf{s_2} = 8$$
$$x_1, x_2, x_3, \mathbf{s_1}, \mathbf{s_2} \geq 0$$

variables are needed in the constraints to transform them into solvable equalities with one definite answer. After the slack variables are introduced, the tableau can be set up to check for optimality as described in Step 3.

Step 3: Setting up the Table

A Simplex table is used to perform row operations on the linear programming model as well as to check a solution for optimality. The table consists of the coefficient corresponding to the linear constraint variables and the coefficients of the objective function. In the table below, the bolded top row of the table states what each column

$$Maximize : z = 8x_1 + 10x_2 + 7x_3$$
$$s.t. : x_1 + 3x_2 + 2x_3 + s_1 = 10$$
$$x_1 + 5x_2 + x_3 + s_2 = 8$$

represents. The following two rows represent the linear constraint variable coefficients from the linear programming model, and the last row represents the objective function variable coefficients.

x1	x2	x3	s1	s2	z	b
1	3	2	1	0	0	10
1	5	1	0	1	0	8
-8	-10	-7	0	0	1	0

Once the table has been completed, the model can be checked for an optimal solution as shown in Step 4.

Step 4: Check Optimality

The optimal solution of a maximization linear programming model are the values assigned to the variables in the objective function to give the largest zeta value. The optimal solution would exist on the corner points of the graph of the entire model. To check optimality using the table, all values in the last row must contain values greater than or equal to zero. If a value is less than zero, it means that variable has not reached its optimal value. As seen in the previous table, three negative values exists in the bottom row indicating that this solution is not optimal. If a table is not optimal, the next step is to identify the pivot variable to base a new table on, as described in Step 5.

Step 5: Identify Pivot Variable

The pivot variable is used in row operations to identify which variable will become the unit value and is a key factor in the conversion of the unit value. The pivot variable can be identified by looking at the bottom row of the tableau and the indicator. Assuming that the solution is not optimal, pick the smallest negative value in the bottom row. One of the values lying in the column of this value will be the pivot variable. To find the indicator, divide the beta values of the linear constraints by their corresponding values from the column containing the possible pivot variable. The intersection of the row with the smallest non-negative indicator and the smallest negative value in the bottom row will become the pivot variable.

In the example shown below, -10 is the smallest negative in the last row. This will designate the x_2 column to contain the pivot variable. Solving for the indicator gives us a value of $\frac{10}{3}$ for the first constraint, and a value of $\frac{8}{5}$ for the second constraint. Due to $\frac{8}{5}$ being the smallest non-negative indicator, the pivot value will be in the second row and have a value of 5.

x1	x2	x3	s1	s2	z	b	Indicator
1	3	2	1	0	0	10	10/3
1	5	1	0	1	0	8	8/5
-8	-10	-7	0	0	1	0	

Smallest Value

Now that the new pivot variable has been identified, the new table can be created in Step 6 to optimize the variable and find the new possible optimal solution.

Step 6: Create the New Table

The new table will be used to identify a new possible optimal solution. Now that the pivot variable has been identified in Step 5, row operations can be performed to optimize the pivot variable while keeping the rest of the table equivalent.

I. To optimize the pivot variable, it will need to be transformed into a unit value (value of 1). To transform the value, multiply the row containing the pivot variable by the reciprocal of the pivot value. In the example below, the pivot variable is originally 5, so multiply the entire row by $\frac{1}{5}$.

x1	x2	x3	s1	s2	z	b	
1/5	1	1/5	0	1/5	0	8/5	Pivot row

II. After the unit value has been determined, the other values in the column containing the unit value will become zero. This is because the x_2 in the second constraint is being optimized, which requires x_2 in the other equations to be zero.

x1	x2	x3	s1	s2	z	b	
	0						
1/5	1	1/5	0	1/5	0	8/5	Pivot row
	0						
	Pivot Column						

In order to keep the table equivalent, the other variables not contained in the pivot column or pivot row must be calculated by using the new pivot values. For each new value, multiply the negative of the value in the old pivot column by the value in the new pivot row that corresponds to the value being calculated. Then add this to the old value from the old tableau to produce the new value for the new table. This step can be condensed into the equation on the next page:

New table value = (Negative value in old tableau pivot column) x (value in new table pivot row) + (Old table value)

Old Tableau:

x1	x2	x3	s1	s2	z	b
1	3	2	1	0	0	10
1	5	1	0	1	0	8
-8	-10	-7	0	0	1	0
	Old pivot column					

New Tableau:

x1	x2	x3	s1	s2	z	b	
2/5	0	7/5	1	-3/5	0	26/5	
1/5	1	1/5	0	1/5	0	8/5	New pivot row
-6	0	-5	0	2	1	16	

Numerical examples are provided below to help explain this concept a little better.

Numerical examples:

I. *To find the s_2 value in row 1:*
New tableau value = (Negative value in old tableau pivot column) * (value in new tableau pivot row) + (Old tableau value)

New tableau value = (-3) * ($\frac{1}{5}$) + 0 = -$\frac{3}{5}$

II. *To find the x_1 variable in row 3:*
New tableau value = (Negative value in old tableau pivot column) * (value in new tableau pivot row) + (Old tableau value)

New value = (10) * ($\frac{1}{5}$) + -8 = -6

Once the new tableau has been completed, the model can be checked for an optimal solution.

Step 7: Check Optimality

As explained in Step 4, the optimal solution of a maximization linear programming model are the values assigned to the variables in the objective function to give the largest zeta value. Optimality will need to be checked after each new tableau to see if a new pivot variable needs to be identified. A solution is considered optimal if all values in the bottom row are greater than or equal to zero. If all values are greater than or equal to zero, the solution is considered optimal and Steps 8 through 11 can be ignored. If negative values exist, the solution is still not optimal and a new pivot point will need to be determined which is demonstrated in Step 8.

Step 8: Identify New Pivot Variable

If the solution has been identified as not optimal, a new pivot variable will need to be determined. The pivot variable was introduced in Step 5 and is used in row operations to identify which variable will become the unit value and is a key factor in the conversion of the unit value. The pivot variable can be identified by the intersection of the row with the smallest non-negative indicator and the smallest negative value in the bottom row.

x1	x2	x3	s1	s2	z	b	Indicator
2/5	0	7/5	1	−3/5	0	26/5	(26/5) / (2/5) = 13
1/5	1	1	0	1/5	0	8/5	(8/5) / (1/5) = 8
-6	0	-5	0	2	1	0	

Smallest Value

With the new pivot variable identified, the new tableau can be created in Step 9.

Step 9: Create New Table

After the new pivot variable has been identified, a new table will need to be created. Introduced in Step 6, the tableau is used to optimize the pivot variable while keeping the rest of the table equivalent.

I. Make the pivot variable 1 by multiplying the row containing the pivot variable by the reciprocal of the pivot value. In the tableau below, the pivot value was $\frac{1}{5}$, so everything is multiplied by 5.

x1	x2	x3	s1	s2	z	b
1	5	1	0	1	0	8

II. Next, make the other values in the column of the pivot variable zero. This is done by taking the negative of the old value in the pivot column and multiplying it by the new value in the pivot row. That value is then added to the old value that is being replaced.

x1	x2	x3	s1	s2	z	b
0	-2	1	1	-1	0	2
1	5	1	0	1	0	8
0	30	1	0	8	1	64

Step 10: Check Optimality

Using the new table, check for optimality. Explained in Step 4, an optimal solution appears when all values in the bottom row are greater than or equal to zero. If all values are greater than or equal to zero, skip to Step 12 because optimality has been reached. If negative values still exist, repeat steps 8 and 9 until an optimal solution is obtained.

Step 11: Identify Optimal Values

Once the table is proven optimal the optimal values can be identified. These can be found by distinguishing the basic and non-basic variables. A basic variable can be classified to have a single 1 value in its column and the rest be all zeros. If a variable does not meet this criteria, it is considered non-basic. If a variable is non-basic it means the optimal solution of that variable is zero. If a variable is basic, the row that contains the 1 value will correspond to the beta value. The beta value will represent the optimal solution for the given variable.

x1	x2	x3	s1	s2	z	b
0	-2	1	1	-1	0	2
1	5	1	0	1	0	8
0	30	1	0	8	1	64

Basic variables: x_1, s_1, z

Non-basic variables: x_2, x_3, s_2

For the variable x_1, the 1 is found in the second row. This shows that the optimal x_1 value is found in the second row of the beta values, which is 8.

Variable s_1 has a 1 value in the first row, showing the optimal value to be 2 from the beta column. Due to s_1 being a slack variable, it is not actually included in the optimal solution since the variable is not contained in the objective function.

The zeta variable has a 1 in the last row. This shows that the maximum objective value will be 64 from the beta column.

The final solution shows each of the variables having values of:

$x_1 = 8$	$s_1 = 2$
$x_2 = 0$	$s_2 = 0$
$x_3 = 0$	$z = 64$

The maximum optimal value is 64 and found at (8, 0, 0) of the objective function.

Conclusion

The Simplex method is an approach for determining the optimal value of a linear program by hand. The method produces an optimal solution to satisfy the given constraints and produce a maximum zeta value. To use the Simplex method, a given linear programming model needs to be in standard form, where slack variables can then be introduced. Using the tableau and pivot variables, an optimal solution can be reached. From the example worked throughout this document, it can be determined that the optimal objective value is 64 and can be found when x_1=8, x_2=0, and x_3=0.

Duality Theorem

The Duality Theorem is a fundamental concept in optimization, particularly in linear programming. It establishes a relationship between two related optimization problems: the primal problem and the dual problem. Here's a concise overview:

Primal Problem

This is the original optimization problem you are trying to solve. A standard form for a linear programming (LP) primal problem is:

$\text{Maximize } c^T x$
subject to
$Ax \leq b$
$x \geq 0$

where x is the vector of variables to be determined, c is the coefficient vector for the objective function, A is the matrix of coefficients for the constraints, and b is the constraint boundary vector.

Dual Problem

From the primal problem, we can construct a related problem known as the dual problem. The dual of the primal maximization problem is a minimization problem:

$\text{Minimize } b^T y$
subject to
$A^T y \geq c$
$y \geq 0$

where y is the vector of dual variables.

Duality Theorem

The Duality Theorem states that:

1. **Weak Duality:** The objective value of any feasible solution to the dual problem provides an upper bound to the objective value of any feasible solution to the primal problem (for maximization). Conversely, for a minimization primal problem, any feasible solution to the dual problem provides a lower bound.

2. **Strong Duality:** If both the primal and dual problems have feasible solutions, then both have optimal solutions, and the optimal objective values are equal. This means that the maximum value of the primal problem is equal to the minimum value of the dual problem.

Implications

- **Feasibility:** If the primal problem has an optimal solution, then the dual problem also has an optimal solution, and vice versa.
- **Optimal Solutions:** The values of the primal and dual solutions are equal at optimality.
- **Complementary Slackness:** This provides a condition that relates the optimal solutions of the primal and dual problems. For each pair of primal and dual constraints, either the primal constraint is tight (the inequality holds as an equality) or the corresponding dual variable is zero.

Example

Consider the following primal problem:

$\text{Maximize}\ 3x_1 + 2x_2$
subject to
$x_1 + x_2 \leq 4$
$x_1 \leq 2$
$x_1, x_2 \geq 0$

The corresponding dual problem would be:

$\text{Minimize}\ 4y_1 + 2y_2$
subject to
$y_1 + y_2 \geq 3$
$y_1 \geq 2$
$y_1, y_2 \geq 0$

By solving these, you can verify that the optimal values of the objective functions of the primal and dual problems are equal.

Non-Linear Programming

Non-linear programming (NLP) is a branch of mathematical optimization that deals with problems in which the objective function or the constraints are non-linear. Unlike linear programming, where the relationships between variables are linear, non-linear programming allows for a broader range of functions, making it more versatile but also more complex to solve.

Key Concepts in Non-Linear Programming

Objective Function: This is the function to be maximized or minimized. In non-linear programming, this function can take various non-linear forms, such as quadratic, exponential, or trigonometric.

Constraints: These are the conditions that the solution must satisfy. Constraints in non-linear programming can also be non-linear equations or inequalities.

Feasible Region: The set of all possible points that satisfy the constraints. The optimal solution to the NLP problem must lie within this region.

Local vs. Global Optima: In non-linear programming, an objective function can have multiple local optima (points where the function value is higher or lower than at nearby points). The global optimum is the best overall solution within the feasible region.

Convexity: A function is convex if a line segment between any two points on the graph of the function lies above or on the graph. Convex functions have the property that any local minimum is also a global minimum, which simplifies optimization.

Solution Methods

Gradient Descent: This is an iterative method for finding the minimum of a function. It involves moving in the direction of the negative gradient (the direction of steepest descent) until convergence.

Newton's Method: An iterative method that uses second-order derivatives (the Hessian matrix) to find the minimum or maximum of a function. It converges faster than gradient descent but is more computationally intensive.

Conjugate Gradient Method: An optimization algorithm that can be more efficient than gradient descent for large-scale problems. It doesn't require the computation of second-order derivatives.

Interior-Point Methods: These methods solve NLP problems by starting from a point within the feasible region and iteratively moving towards the boundary while improving the objective function.

Penalty Methods: These methods convert a constrained problem into a series of unconstrained problems by adding a penalty term to the objective function for violating constraints.

Kuhn-Tucker Conditions

The Kuhn-Tucker conditions, also known as the Karush-Kuhn-Tucker (KKT) conditions, are a set of necessary conditions for a solution in nonlinear programming to be optimal. They are used in mathematical optimization for problems with inequality constraints. The conditions extend the method of Lagrange multipliers, which is used for problems with equality constraints.

Here's a brief overview of the Kuhn-Tucker conditions:

Problem Formulation

Consider the following nonlinear programming problem:

$\min f(x)$

subject to

$g_i(x) \leq 0 \quad \text{for } i = 1, \ldots, m$

$h_j(x) = 0 \quad \text{for } j = 1, \ldots, p$

where $f(x)$ is the objective function, $g_i(x)$ are the inequality constraint functions, and $h_j(x)$ are the equality constraint functions.

Kuhn-Tucker Conditions

For a point x^* to be optimal, there must exist multipliers λ_i (for inequality constraints) and μ_j (for equality constraints) such that the following conditions hold:

1. **Stationarity (Gradient Condition):**
   $\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{j=1}^{p} \mu_j \nabla h_j(x^*) = 0$

2. **Primal Feasibility:**
   $g_i(x^*) \leq 0 \quad \text{for } i = 1, \ldots, m$
   $h_j(x^*) = 0 \quad \text{for } j = 1, \ldots, p$

3. **Dual Feasibility:**
   $\lambda_i \geq 0 \quad \text{for } i = 1, \ldots, m$

4. **Complementary Slackness:**
   $\lambda_i g_i(x^*) = 0 \quad \text{for } i = 1, \ldots, m$

UNIT V

ECONOMIC DYNAMICS

Differential equations

Differential equations are mathematical equations that involve functions and their derivatives. They describe the relationship between a function and its rate of change, making them essential for modeling various physical, biological, and economic systems. Here is a more detailed explanation:

Types of Differential Equations

1. Ordinary Differential Equations (ODEs): These involve functions of a single variable and their derivatives. They are typically written as:

$$\frac{dy}{dx} = f(x, y)$$

where y is the dependent variable, x is the independent variable, and f is a given function.

2. **Partial Differential Equations (PDEs)**: These involve functions of multiple variables and their partial derivatives. An example is:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

which is the heat equation, where u is the dependent variable, and t and x are the independent variables.

Order and Degree

- **Order**: The order of a differential equation is the highest derivative present. For example, $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$ is a second-order ODE.
- **Degree**: The degree is the power of the highest order derivative, provided the equation is polynomial in derivatives. For example, $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 = y$ is of second order and degree 2.

Solutions to Differential Equations

- **General Solution**: The general solution of a differential equation contains arbitrary constants and represents a family of functions.
- **Particular Solution**: A particular solution is obtained by assigning specific values to the arbitrary constants in the general solution, often based on initial or boundary conditions.

Methods of Solving Differential Equations

1. **Separation of Variables**: Used for ODEs that can be expressed as $\frac{dy}{dx} = g(x)h(y)$, where the variables can be separated and integrated separately.
2. **Integrating Factor**: Used for linear first-order ODEs of the form $\frac{dy}{dx} + p(x)y = q(x)$. The integrating factor, $\mu(x) = e^{\int p(x)dx}$, transforms the equation into an exact differential equation.
3. **Characteristic Equation**: Used for solving linear differential equations with constant coefficients. For example, for $ay'' + by' + cy = 0$, the characteristic equation $ar^2 + br + c = 0$ helps find the general solution.
4. **Numerical Methods**: For complex differential equations where analytical solutions are difficult, numerical methods like Euler's method, Runge-Kutta methods, and finite

Applications of Differential Equations

Physics: Describing motion (Newton's second law), heat conduction (heat equation), wave propagation (wave equation).

Biology: Modeling population dynamics, the spread of diseases, and biochemical reactions.

Economics: Modeling growth rates, decay processes, and dynamic systems.

Engineering: Describing systems in control theory, electrical circuits, and fluid dynamics.

Understanding and solving differential equations are crucial skills in many scientific and engineering disciplines, as they provide a framework for modeling and analyzing dynamic systems.

Solow's Model

Introduction:

Prof. Robert M. Solow made his model an alternative to Harrod-Domar model of growth.

It ensures steady growth in the long run period without any pitfalls. Prof. Solow assumed that Harrod-Domar's model was based on some unrealistic assumptions like fixed factor proportions, constant capital output ratio etc.

Solow has dropped these assumptions while formulating its model of long-run growth. Prof. Solow shows that by the introduction of the factors influencing economic growth, Harrod-Domar's Model can be rationalised and instability can be reduced to some extent.

He has shown that if technical coefficients of production are assumed to be variable, the capital labour ratio may adjust itself to equilibrium ratio in course of time.

In Harrod-Domar's model of steady growth, the economic system attains a knife-edge balance of equilibrium in growth in the long-run period.

This balance is established as a result of pulls and counter pulls exerted by natural growth rate (Gn) (which depends on the increase in labour force in the absence of technical changes) and warranted growth rate (Gw) (which depends on the saving and investment habits of household and firms).

However, the key parameter of Solow's model is the substitutability between capital and labour. Prof. Solow demonstrates in his model that, "this fundamental opposition of warranted and natural rates turns out in the end to flow from the crucial assumption that production takes place under conditions of fixed proportions.”

The knife edge balance established under Harrodian steady growth path can be destroyed by a slight change in key parameters.

Prof. Solow retains the assumptions of constant rate of reproduction and constant saving ratio etc. and shows that substitutability between capital and labour can bring equality between warranted growth rate (Gw) and natural growth rate (Gn) and economy moves on the equilibrium path of growth.

In other words, according to Prof. Solow, the delicate balance between Gw and Gn depends upon the crucial assumption of fixed proportions in production. The knife edge equilibrium between Gw and Gn will disappear if this assumption is removed. Solow has provided solution to twin problems of disequilibrium between Gw and Gn and the instability of capitalist system.

In short, Prof. Solow has tried to build a model of economic growth by removing the basic assumptions of fixed proportions of the Harrod-Domar model. By removing this assumption, according to Prof. Solow, Harrodian path of steady growth can be freed from instability. In this way, this model admits the possibility of factor substitution.

Assumptions:

Solow’s model of long run growth is based on the following assumptions:

The production takes place according to the linear homogeneous production function of first degree of the form

$Y = F(K, L)$

Y = Output

K = Capital Stock

L = Supply of labour force

1. The above function is neo-classic in nature. There is constant returns to scale based on capital and labour substitutability and diminishing marginal productivities. The constant returns to scale means if all inputs are changed proportionately, the output will also change proportionately. The production function can be given as aY = F (aK, al)

2. The relationship between the behaviour of savings and investment in relation to changes in output. It implies that saving is the constant fraction of the level of output. In this way, Solow adopts the Harrodian assumption that investment is in direct and rigid proportion to income.

In symbolic terms, it can be expressed as follows:

$I = dk/ dt = sY$

Where

S—Propensity to save.

K—Capital Stock, so that investment I is equal

3. The growth rate of labour force is exogenously determined. It grows at an exponential rate given by

$L = L0 ent$

Where L—'Total available supply of labour.

n—Constant relative rate at which labour force grows.

4. There is full employment in the economy.

5. The two factors of production are capital and labour and they are paid according to their physical productivities.

6. Labour and capital are substitutable for each other.

7. Investment is not of depreciation and replacement charges.

8. Technical progress does not influence the productivity and efficiency of labour.

9. There is flexible system of price-wage interest.

10. Available capital stock is fully utilized.

Following these above assumptions, Prof. Solow tries to show that with variable technical co-efficient, capital labour ratio will tend to adjust itself through time towards the direction of equilibrium ratio. If the initial ratio of capital labour ratio is more, capital and output will grow more slowly than labour force and vice-versa.

To achieve sustained growth, it is necessary that the investment should increase at such a rate that capital and labour grow proportionately i.e. capital labour ratio is maintained.

Solow's model of long-run growth can be explained in two ways:

A. Non-Mathematical Explanation.

B. Mathematical Explanation.

A. Non-Mathematical Explanation:

According to Prof. Solow, for attaining long run growth, let us assume that capital and labour both increase but capital increases at a faster rate than labour so that the capital labour ratio is high. As the capital labour ratio increases, the output per worker declines and as a result national income falls.

The savings of the community decline and in turn investment and capital also decrease. The process of decline continues till the growth of capital becomes equal to the growth rate of labour. Consequently, capital labour ratio and capital output ratio remain constant and this ratio is popularly known as "Equilibrium Ratio".

Prof. Solow has assumed technical coefficients of production to be variable, so that the capital labour ratio may adjust itself to equilibrium ratio. If the capital labour ratio is larger than equilibrium ratio, than that of the growth of capital and output capital would be lesser than labour force. At some time, the two ratios would be equal to each other.

In other words, this is the steady growth, according to Prof. Solow as there is the steady growth there is a tendency to the equilibrium path. It must be noted here that the capital-labour ratio may be either higher or lower.

Like other economies, Prof. Solow also considers that the most important feature of an underdeveloped economy is dual economy. This economy consists of two sectors-capital sector or industrial sector and labour sector or agricultural sector. In industrial sector, the rate of accumulation of capital is more than rate of absorption of labour.

With the help of variable technical coefficients many employment opportunities can be created. In agricultural sector, real wages and productivity per worker is low. To achieve sustained growth, the capital labour ratio must be high and underdeveloped economies must follow Prof. Solow to attain the steady growth.

This model also exhibits the possibility of multiple equilibrium positions. The position of unstable equilibrium will arise when the rate of growth is not equal to the capital labour ratio. There are other two stable equilibrium points with high capital labour ratio and the other with low capital labour ratio.

If the growth process starts with high capital labour ratio, then the development variables will move in forward direction with faster speed and the entire system will grow with high rate of growth. On the other hand, if the growth process starts with low capital labour ratio then the development variables will move in forward direction with lesser speed.

To conclude the discussion, it is said that high capital labour ratio or capital intension is very beneficial for the development and growth of capitalist sector and on the contrary, low capital-labour ratio or labour-intensive technique is beneficial for the growth of labour sector.

B. Mathematical Explanation:

This model assumes the production of a single composite commodity in the economy. Its rate of production is Y (t) which represents the real income of the community. A part of the output is consumed and the rest is saved and invested somewhere.

The proportion of output saved is denoted by s. Therefore, the rate of saving would be sY (t). The capital stock of the community is denoted by K it). The rate of increase in capital stock is given by dk/dt and it gives net investment.

Since investment is equal to saving so we have following identity:

K = sY … (1)

Since output is produced by capital and labour, so the production function is given by

Y = F (K, L) … (2)

Putting the value of Y from (2) in (1) we get

S = s F (K, L) … (3)

Where

L is total employment

F is functional relationship

Equation (3) represents the supply side of the system. Now we are to include demand side too. As a result of exogenous population growth, the labour force is assumed to grow at a constant rate relative to n. Thus,

L (t) = L0ent … (4)

Where

L—Available supply of labour

Putting the value of L in equation (3) we get

K = sF (K, L0ent) …(5)

The right hand of the equation (4) shows the rate of growth of labour force from period o to t or it can be regarded as supply curve for labour.

"It says that the exponentially growing labour force is offered for employment completely in elastically. The labour supply curve is a vertical line, which shifts to the right in time as the labour force grows. Then the real wage rate adjusts so that all available labour is employed and the marginal productivity equation determines the wage rate which will actually rule."

If the time path of capital stock and of labour force is known, the corresponding time path of real output can be computed from the production function. Thus, the time path of real wage rate is calculated by marginal productivity equation.

The process of growth has been explained by Prof. Solow as, "At any moment of time the available labour supply is given by (4) and available stock of capital is also a datum. Since the real return to factors will adjust to bring about full employment of labour and capital we can use the production function (2) to find the current rate of output. Then the propensity to save tells us how much net output will be saved and invested. Hence, we know the net accumulation of capital during the current period. Added to the already accumulated stock this gives us the capital available for the next period and the whole process can be repeated."

Possible Growth Patterns:

To find out whether there is always a capital accumulation path consistent with any rate of growth of labour force, we should know the accurate shape of production function otherwise we cannot find the exact solution.

For this, Solow has introduced a new variable:

$$r = \frac{K}{L}$$

Where K/L Capital Labour Ratio

$$K = rL$$

But $$L = L_0 e^{nt}$$

$$K = rL_0 e^{nt}$$

Differentiating with respect to t we get

$$\frac{dk}{dt} = nrL_0 e^{nt} + L_0 e^{nt} \frac{dr}{dt}$$

$$\frac{dk}{dt} = \left(nr + \frac{dr}{dt}\right) L_0 e^{nt}$$

Substituting this value in equation (5) we get ...(5)

$$\left(nr + \frac{dr}{dt}\right) L_0 e^{nt} = sF\,(K, L_0 e^{nt})$$

or $$\left(nr + \frac{dr}{dt}\right) L_0 e^{nt} = sFL_0 e^{nt} \frac{K}{L_0 e^{nt}}, 1$$

or $$nr + \frac{dr}{dt} = sF\left(\frac{K}{L_0 e^{nt}}, 1\right)$$

Since $$\frac{K}{L_0 e^{nt}} = r$$

$$nr + \frac{dr}{dt} = sF\,(r, 1)$$

$$\frac{dr}{dt} = sF\,(r, 1) — nr$$

or $$r = sF\,(r, 1) — nr$$

Where $$r = K/L$$ ...(6)

n = relative share of change of labour force (i/1)

The function F(r, 1) gives output per worker or it is the total product curve as varying amounts ‘r’ of capital are employed with one unit of labour. The equation (6) states that, “the rate of change of the capital labour ratio as the difference of two terms, one representing the increment of capital and one the increment of labour.”

The diagrammatic representation of the above growth pattern is as under:

In diagram 1, the line passing through origin is nr. The total productivity curve is the function of SF (r, 1) and this curve is convex to upward. The implication is that to make the output positive it must be necessary that input must also be positive i.e. diminishing marginal productivity of capital. At the point, of intersection i.e. nr = sf (r, 1) and r’ = o when r’ = o then capital labour ratio corresponds to point r* is established.

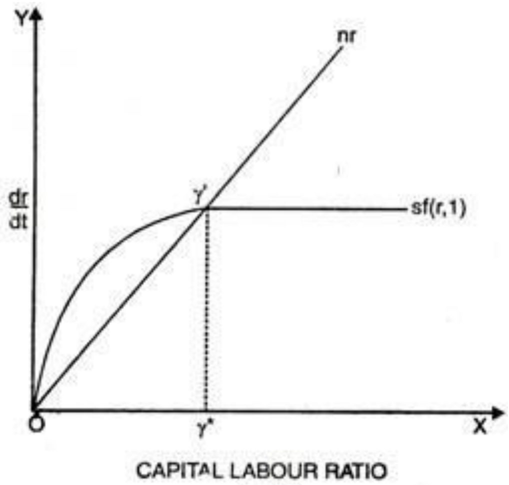


Fig. 1

Now capital and labour will grow proportionately. Since Prof. Solow considers constant returns to scale, real output will grow at the same rate of n and output per head of labour, force will remain constant.

In mathematical terms, it can be explained as:

$$\frac{dr/dt}{t} = \frac{dK/dt}{K} - \frac{dL/dt}{L}$$

Now $$\frac{dL/dt}{L} = n$$

$$\frac{dk}{dt} = sF\,(K, L)$$

$$\frac{dr}{dt} = r\,\frac{sF\,(K\,,\,L)}{K} - nr$$

Since it was assumed to have constant returns to scale.

$$\frac{dr}{dt} = r\,sF\left(1\,\frac{L}{K}\right) - nr$$

$$= \frac{rsF\,(K/L\,,\,1)}{K/L} - nr$$

$$= \frac{rsF\,(r, i)}{K/L} - nr$$

Thus $$\frac{dr}{dt} = sF\,(r, 1) - nr$$

Which is the same as $= n$ (2)

Path of Divergence:

Here we are to discuss the behaviour of capital labour ratio, if there is divergence between r and r". There are two cases:

(i) When r > r*

(ii) When r < r*

If r > r* then we are towards the right of intersection point. Now nr > sF (r, 1) and from equation (6) it is easily shown that r will decrease to r*. On the other hand if we move towards left of the intersection point where nr < sF (r, 1), r>o and r will increase towards r*. Thus, equilibrium will be established at point E and sustained growth will be achieved. Thus, the equilibrium value of r* is stable.

According to Prof. Solow, “Whatever the initial value of the capital labour ratio, the system will develop towards a state of balanced growth at a natural rate. If the initial capital stock is below the equilibrium ratio, capital and output will grow at a faster rate than the labour force until the equilibrium ratio is approached. If the initial ratio is above the equilibrium value, capital and output will grow more slowly than the labour force. The growth of output is always intermediate between those of labour and capital.”

The stability depends upon the shape of the productivity curve sF(r, 1) and it is explained with the help of a diagram given below:

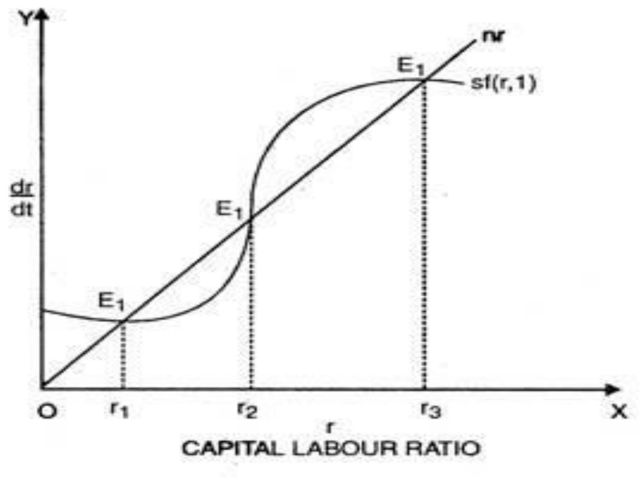


Fig. 2

In the figure 2 the productivity curve sf (r, 1) intersects the ray nr at three different points E1 , E2 , E3. The corresponding capital labour ratio is r1, r2 and r3. The points are r3 stable but r2 is not stable. Taking point r1 first if we move slightly towards right nr > sf(r, 1) and r is negative implying that r decreases.

Thus, it has a tendency to slip back to r1 .If we move slightly towards its left nr < sf (r, 1) and r is positive which shows that r increases and there is a tendency to move upto point r1. Therefore, a slight movement away from r1 creates conditions that forces a movement towards showing that r1 is a point of stable equilibrium.

Likewise, we can show that r3 is also a point of stable equilibrium. If we move slightly towards right of r2, sf (r, 1) nr and r is positive and there is a tendency to move away from r2.

On the other hand, if we move slightly towards left of r2 nr > sf (r, 1) so that r is negative and it has a tendency to slip downwards towards r1. Therefore depending upon initial capital labour ratio, the system will develop to balanced growth at capital labour ratio r1 and r3. If the initial ratio is between o and r2, the equilibrium is at r1 and if the ratio is higher than r2 then equilibrium is at r3.

To conclude Solow puts, "When production takes place under neoclassical conditions of variable proportions and constant returns to scale, no simple opposition between natural and warranted rates of growth is possible. There may not be any knife edge. The system can adjust to any given rate of growth of labour force and eventually approach a state of steady proportional expansion" i.e.

$$\Delta K/K = \Delta L/L = \Delta Y/Y$$

Merits of the Model:

Solow's growth model is a unique and splendid contribution to economic growth theory. It establishes the stability of the steady-state growth through a very simple and elementary adjustment mechanism.

Certainly, the analysis is definitely an improvement over Harrod-Domar model, as he succeeded in demonstrating the stability of the balanced equilibrium growth by implying neo-classical ideas. In fact, Solow' growth model marks a brake through in the history of economic growth.

The merits of Prof. Solow's model are under-mentioned:

(i) Being a pioneer of neo-classical model, Solow retains the main features of Harrod-Domar model like homogeneous capital, a proportional saving function and a given growth rate in the labour forces.

(ii) By introducing the possibility of substitution between labour and capital, he gives the growth process and adjustability and gives more realistic touch.

(iii) He considers a continuous production function in analysing the process of growth.

(iv) Prof. Solow demonstrates the steady-state growth paths.

(v) He successfully shunted aside all the difficulties and rigidities of modern Keynesian income analysis.

(vi) The long-run rate of growth is determined by an expanding labour force and technical process.

Short Comings of the Model:

1. No Study of the Problem of Balance between G and Gw:

Solow takes up only the problem of balance between warranted growth (Gw) and natural growth (Gn) but it does not take into account the problem of balance between warranted growth and the actual growth (G and Gw).

2. Absence of Investment Function:

There is a absence of investment function in Solow's model and once it is introduced, problem of instability will immediately reappear in the model as in the case of Harrodian model of growth.

3. Flexibility of Factor Price may bring Certain Problems:

Prof. Solow assumed the flexibility of factor prices but it may bring certain difficulties in the path of steady growth.

For example, the rate of interest may be prevented from falling below a certain minimum level and this may in turn, prevent the capital output ratio from rising to a level necessary for sustained growth.

4. Unrealistic Assumptions:

Solow's model is based on the unrealistic assumption that capital is homogeneous and malleable. But capital goods are highly heterogeneous and may create the problem of aggregation. In short, it is not easy to arrive at the path of steady growth when there are varieties of capital goods in the market.

5. No Study of Technical Progress:

This model has left the study of technological progress. He has merely treated it as an exogenous factor in the growth process. He neglects the problem of inducing technical progress through the process of learning, investment and capital accumulation.

6. Ignores the Composition of Capital Stock:

Another defect of Prof. Solow's model is that it totally ignores the problem of composition of capital stock and assumes capital as a homogeneous factor which is unrealistic in the dynamic world of today. Prof. Kaldor has forged a link between the two by making learning a function of investment.

Harrod-Domar Model of Economic Growth

Capital formation plays a very important role in the process of development of a country. According to the Harrod-Domar model, economic growth depends on two important factors, viz., the saving ratio (i.e., the percentage of national income saved per annum) and the capital-output ratio.

Since the capital-output ratio remains constant in the short run, the rate of growth of a nation depends largely on the rate of saving. A country which has the capacity to save and invest at least 20 to 25% of its national income will be able to achieve a satisfactory growth rate of 5 to 6% per annum.

In the words of Daniel Fusfeld, "The modern economy is a gigantic mechanism for the generation of an economic surplus and the accumulation of capital. In a modern economy, the surplus is used to increase output. It is transformed into capital goods and knowledge (technology) that raise the productive potential of the economy."

In the Harrod-Domar model the rate of growth of an economy (g) is expressed as:

$$g = (s/v) \times 100\%$$

where s is the saving ratio and v is the incremental capital-output ratio. If s = 10 % and v = 3⅓, g will be 3%. This implies that if 100 units of capital are required to produce 33 units of output in a country and the country can save and invest 10% of its national income per annum, it can achieve per capita income growth of 3% per annum. If s increases to 20% g will be 6%, provided v remains constant. The Harrod-Domar model may now be discussed in detail.

Saving, Investment and Growth: the Harrod-Domar Model:

The problem of aggregate supply and demand in the long run is compli-cated by the dual role of investment. Investment creates demand — via the multiplier, just as it creates supply — adding to society's productive capac-ity. The question is: which aspect is more important if we consider a long period of time.

This problem led, in the period after Keynes' General Theory, to a number of attempts to make that theory dynamic, i.e., to enable us to predict not only national income in a particular period but also its path of change over time. An example of this kind of approach is the celebrated Harrod-Domar model.

This theory was developed independently by Sir Roy Harrod and Evsey Domar.

The theory involves an examination of the following equation, in which Y stands for annual national income (or output), AY for a year's increase in national income, I for annual investment and S for annual savings:

$$\frac{\Delta Y}{Y} = \frac{\Delta Y}{I} = \frac{S}{Y}$$

By making certain assumptions, we can use this equation to show some of the difficulties of keeping aggregate supply and demand in proper balance in a growing economy. To start with, let us suppose that the fraction of income people wish to save is some fixed number, say, one-tenth.

This is quite a realistic assumption. Although in the short run the marginal propen-sity to save might be expected to rise with income, something else happens in the long run, inasmuch as people have enough time to adjust their living standards to higher levels of income.

We also assume that the amount of machinery and other capital goods used to produce a given level of output remains more or less fixed.

Now, we can argue that the term AY/I, which represents the increase in income in a year divided by the increase in the stock of capital (i.e., invest-ment) in a year, will be some pure number — say 1/3. In other words, business people who expand their plants and equipment by Rs 3 and will have the capacity for producing Re. 1 a year more output than before. Hence, 1/3 is the incremental capital-output ratio.

Now, from these first it is possible to determine a 'rate of growth' for this economy. In equilibrium, the amount that households desire to save will have to be equal to the amount that businesspeople wish to invest, or S = l.

Hence, growth can be expressed as:

$$g = \frac{S}{Y}.\frac{\Delta Y}{I} = \frac{I}{Y}.\frac{\Delta Y}{I} = \frac{\Delta Y}{I} \quad ....(1)$$

ΔY/Y being the increase in output divided by the initial level of output, or the rate of growth of economy. In our example, it will be equal to 1/30 or 3.3%.

ΔY/Y = ΔY/I.S/Y = (1/3) (1/10) = (1/30) = 3.3% … (2)

Problem with model:

However, there is one major problem with the model. The growth rate that keeps investment and saving happily in balance (sometimes called the equilibrium or warranted rate of growth) may be quite different from the rate at which population is growing, called the natural rate of growth.

In this theory, there really is no guarantee at all that aggregate supply and aggregate demand will grow in harmony over time. On the contrary, the system is for ever poised toward runaway inflations or depressions, called knife-edge instability.

So, in this model the economy more on a razor's edge, or misstep in either direction being fatal. However, while discussing the 'knife-edge' properties of steady-state growth path Harrod argued that potential accelerator and the saving rate that drove the actual growth rate back the warranted growth rate every time it deviated from it.

Usefulness of the model:

The model highlights certain important points, viz., saving leads to an increase in investment, which leads to an increase in income (through the incremental capital/output ratio), which leads to more saving, more investment and more income.

Capital accumulation, expansion of labour force and technical progress are given specific roles by Harrod in his model, but he also examined the role of expectations and possibilities of instability arising there-from. He has brought into focus the fundamental economics of growth, the necessary relations existing between dynamic elements — population change, technological progress and long-term sav-ing — of an advancing society. He also emphasised the dual character of investment — it generates income and adds to the productive capacity of the economy.

Harrod and Domar were particularly concerned with the role of invest-ment as capital accumulation and as a component of aggregate demand. Their model incorporated a simple accelerating investment function based on expected real income.

In summary, the Harrod-Domar model considers three basic issues:

1. Whether or not steady-state growth is possible;
2. The probability of steady-state growth at full employment;
3. The stability or otherwise of the warranted rate of growth.

Criticisms:

The Harrod-Domar model neglected the effects of relative prices on factor proportions, implying they were in fixed ratio. So, even though they implied an aggregate production function they escaped the main criticisms of the production function incorporated into the neo-classi-cal growth model.

Criticism of Harrod's theory is generally targeted at his behavioural assumption that producers invest only to meet expected demand in the next time period. This assumption eliminates any long-term investment plans, or anticipation of long-term demand trends, by producers.

A related criticism is that producers are not required to respond to unanticipated demand levels only by varying their output. Price variations, of course, are another option that would be particularly useful in the short run.

Conclusion:

The Harrod-Domar model set the scene for subsequent work on growth as their framework was sufficiently general to incorporate technical progress, money and other effects.

Linear Differential Equations with Constant Coefficients

(First and Second Order)- Applications:

Linear differential equations with constant coefficients are a fundamental topic in the study of differential equations. They have numerous applications in various fields, such as physics, engineering, economics, and more. Here, we'll discuss first and second-order linear differential equations with constant coefficients and their applications.

First-Order Linear Differential Equations

A first-order linear differential equation with constant coefficients has the general form:

$$\frac{dy}{dt} + ay = f(t)$$

where a is a constant and $f(t)$ is a given function of t.

Solution

The general solution to this equation can be found by the method of integrating factors. The integrating factor is given by e^{at}, leading to the solution:

$$y(t) = e^{-at}\left(\int e^{at} f(t)\, dt + C\right)$$

where C is the constant of integration.

Applications

1. **RC Circuits**: In electrical engineering, the charging and discharging of a capacitor in an RC circuit is described by a first-order linear differential equation.

$$\frac{dV}{dt} + \frac{1}{RC}V = \frac{V_0}{RC}$$

Samuelson's Multiplier Accelerator Interaction Model

The principle of acceleration working by itself is perhaps not much forceful but recently it has attained more importance in the trade cycle theory by its alliance with the multiplier principle. Professor Samuelson has built a model of multiplier- accelerator interaction. He could show that the interaction of the accelerator with the multiplier is capable, under certain circumstances, of generating continuous cyclical fluctuations.

After P. A. Samuelson, J. R. Hicks, R. F. Harrod and A. Hansen have made fairly successful attempts to integrate the two concepts and thus introduced remarkable improvements in the theory of economic growth. It is, therefore, quite interesting and useful to analyse the combined (leverage) effects of multiplier and accelerator on national income propagation. The combined effects of autonomous and induced investment are expressed in what Hansen called the Super-Multiplier.

Paul Samuelson has derived the super multiplier as follows:

$Yt = Ct + It$ ….(1)

Income in period t equals the consumption in period t – 1 plus investment in period t

$Yt = bYt\text{-}1 + lt$ ….(2)

where b is the MPC in period t – 1.

Investment in period t is partly autonomous (Ia) and partly induced (Id). The induced investment depends upon changes in consumption which in turn depend upon changes in income. Therefore, we can express as

$It = la+Id$

$= Ia+V(Ct\text{-}Ct\text{–}1)$

$= Ia +V\{bYt\text{-}1\text{-}bYt\text{-}2\}$

$= Ia + V.b(Yt\text{-}1\text{-}Yt\text{-}2)$….(3)

The expression in equation (3) tells us that changes in net investment and hence income for period t can be estimated as a weighted sum, the weights being the values of b (MFC) and V (the output-capital ratio). Professor J.R. Hicks has called the joint action of b and V as the super multiplier and used it to build up his theory of the trade cycle.

$$\text{Super Multiplier } K_b = \frac{1}{1-b-V} = \frac{1}{s-V}$$

That is, the speed with which income increases as a result of accelerator-multiplier interaction depends inversely on the marginal propensity to save and directly on the value of the acceleration coefficient. The lower is the MPS and the higher is the value of the accelerator, the greater the speed with which changes in net investment are multiplied into changes in income. We show the working of the super-multiplier through the Table below given hereunder.

The Interaction:

In the Table below given below, we can easily see the process of income propagation through the multiplier and acceleration interaction. We assume that

(i) MPC = ½

ii) Acceleration Coefficient = 2.

In the first period there is an initial outlay of Rs. 10 crores, which does not lead to any induced investment. Hence, the total rise in national income in the first period is Rs. 10 crores (being equal to the initial outlay of Rs. 10 crores).

Since the MPC =1/2, induced consumption in the second period is Rs. 5 crores (shown in the column 3) and the acceleration coefficient being 2, the induced investment in the second period is Rs. 10 crores, (shown in column 4) and the total leverage effect (total increase in national income) is Rs. 25 crores (shown in column 5).

Similarly, in the third period we get increased consumption of Rs. 12.50 crores and induced net investment of Rs. 15 crores (being the difference between 12.50 crores and 5 crores in column 3). Thus, total income in the fourth period has reached the peak level of Rs. 41.25 crores, as a result of the combined multiplier and acceleration effects i.e. through their interaction also called Super Multiplier. Then, in the fifth period, the marginal income increase starts falling off. It falls to rock bottom level of 1.2 crores in the 8th period and then again starts rising in the 9th period from 2 crores to 12 crores.

Multiplier and Acceleration Interaction Effect on Income
The Leverage Effects (Rs. crores)

1 Multiplier period	2 Initial Outlay consumption	3 Induced Investment (ΔC = Col. 5 × $\frac{1}{2}$)	4 Induced Net in National (ΔC × 2)	5 Total Increase Income 3 + 4
1	Rs. 10	0	0	Rs. 10
2	Rs. 10	Rs. 5	Rs. 10	Rs. 25
3	Rs. 10	Rs. 12.50	Rs. 15	Rs. 37.50
4	Rs. 10	Rs. 18.75	Rs. 12.50	Rs. 41.25
5	Rs. 10	Rs. 20.62	Rs. 3.74	Rs. 34.36
6	Rs. 10	Rs. 17.18	Rs. –4.88	Rs. 22.30
7	Rs. 10	Rs. 11.15	Rs. –12.06	Rs. 9.09
8	Rs. 10	Rs. 4.5.*	Rs. –13.3	Rs. 1.2
9	Rs. 10	Rs. 0.5**	Rs. –8*	Rs. 2
10	Rs.10	Re. 1	Re. 1	Rs. 12
11	Rs. 10	Rs. 6	Rs. 10	Rs. 26

* Rounded off.

** Adapted from K.K. Kurihara, *Monetary Theory and Public Policy*, pp. 232-33.

Assumptions : (*i*) **Marginal Prosperity to Consume = 1/2 = 0.5**
(*ii*) **Acceleration Coefficient = 2**

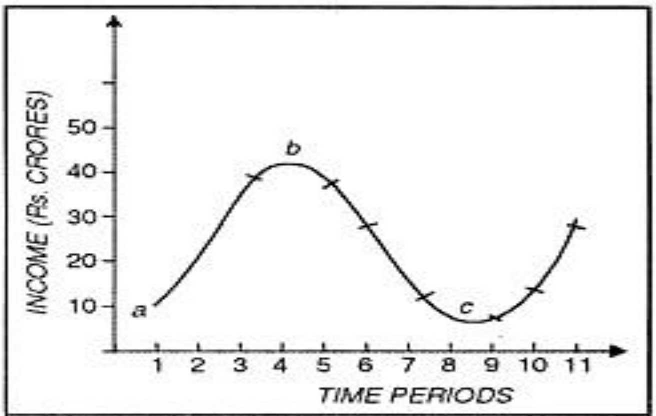
It goes up to 26 crores in the 11th period, thereby completing a cycle. If we go on calculating the multiplier-acceleration effects in the various columns we will find that the result is quite a moderate type of recurring cycle which repeats itself indefinitely. If we show the values of income and the time periods on a graph, we get a regular trade cycle m & n shown in the figure given below.

This shows that an MPC of less than unity gives an answer to the crucial question: Why does the cumulative process come to an end before a complete collapse or before full employment? Hansen says that the rise in income progressively slows down on account of the fact that a large part of the increase in income in each successive period is not spent on consumption. This results in a decline in the volume of induced investment and when such a decline exceeds the increase in induced consumption, a decline in income sets in. "Thus, it is the marginal propensity to save which calls a halt to the expansion on top of the multiplier process."

We have in our table above assumed constant values of multiplier and acceleration coefficients but in a dynamic economy they also vary cyclically. Thus, when we study the results of leverage effects or interaction of multiplier and acceleration coefficients as they vary cyclically, we find that the levels of income will be subject to various types of fluctuations depending on the values of the acceleration and the multiplier.

Professor Paul A. Samuelson attempted to combine different values of the multiplier and accelerator to analyse the nature of income streams generated by them. He found that four different types of fluctuations are obtained when the super multiplier with different values works.

The Different Types of Income Fluctuations:



Accelerator-Multiplier Interaction goes to Generate a Trade Cycle.

In the analysis of the process of multiplier accelerator working together, the values which we assume are of great importance. If the values of K and a are high, income would rise continuously at an ever-increasing rate, and there would be no downturn and therefore no cycle.

The path would be similar to curve A in below figure. At the opposite extreme, low values of the accelerator will also fail to generate a cycle; income will merely converge to the new equilibrium level which would have been achieved by the operation of the Multiplier alone, and will stay there; the accelerator will work to make income grow faster during the initial stage of expansion, but it will not be strong enough to alter the eventual outcome. Such a path of income is exemplified by curve B in below figure. If trade cycles are to be generated, it is necessary that the values of multiplier and accelerator lie between certain upper and lower extremes.

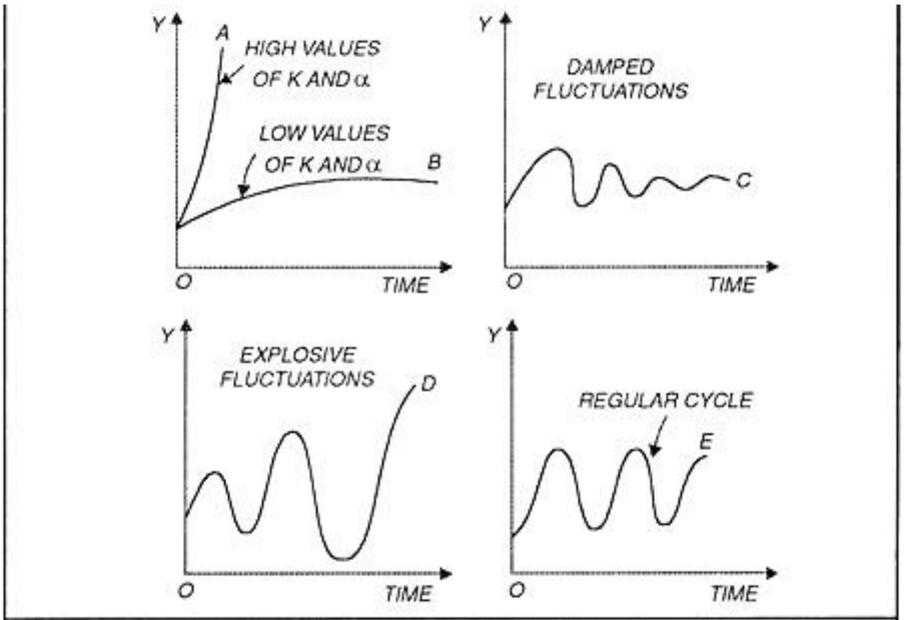


Fig. 11.2 Accelerator-Multiplier Interaction to Produce Fluctuations of Different Types.

Three other distinct types of cycles may be generated according to the relative values of accelerator and multiplier. If accelerator is small relatively to Multiplier the oscillations would be ‘damped’—that is, they would become weaker and weaker and eventually die away altogether leaving income constant at its central value. Curve C in above figure illustrates this kind of fluctuation. High values of Accelerator give rise to explosive fluctuations (Curve D), the cycles becoming stronger and stronger. The third type of cycle is a perfectly regular one exemplified by curve E where the cycle is regular as it is having equal periodicity and amplitude.

Importance:

Thus, we find that the Super-Multiplier (interaction of multiplier and accelerator) assumes great importance in as much as it tends to speed up the rate at which the national income raises through the multiplier effects. In the first place, study of interaction of the two principles has paved the way for a more accurate analysis of the nature of the cyclical processes.

Further, an analysis of the interaction shows that it is possible to explain turning points in business cycle without resorting to special explanations. These factors are: a marginal propensity to consume of less than one plus the acceleration effect, the former being perhaps more important. The acceleration principle, before Keynes, was based on Say's Law wherein increase in investment and a decrease in the rate of consumption would lead to a similar decline in investment—i.e., a cumulative expansion or contraction without limit.

Thus, the pre-Keynesian theory of acceleration gave an exaggerated picture of instability in the economy. But with the introduction of the concept of multiplier and consumption function, the long-sought-for limits to the fluctuations short of zero were at last found. Keynes's concept of consumption function has brought forward the true significance of acceleration principle for business cycle analysis. According to Prof. K. Kurihara, "it is in conjunction with the multiplier analysis based on the concept of the marginal propensity to consume (being less than one) that the acceleration principle serves as a useful tool for business cycle analysis and as a helpful guide to business cycle policy.

Thirdly, it shows that it is a combination of the multiplier and the accelerator which seems to be capable of producing cyclical fluctuations. The multiplier alone produces no cycles from any given impulse. It only gradually increases income to a constant level as determined by the propensity to consume. But if the principle of Acceleration is also introduced, the result is a series of oscillations about what might be called the multiplier level.

The accelerator at first carries total income above this level, but as the rate of increase of income diminishes; the accelerator induces a downturn which carries total income below the multiplier level, then up again, and so on. Professor J R. Hicks has used the accelerator multiplier interaction for building up a new theory of the trade cycle in this context.

Limitations:

A casual student of the super-multiplier (multiplier-acceleration principle) might feel that it is very easy to raise the economy out of the depths of depression simply by having a small increase in autonomous investment. This would stimulate consumption via the multiplier effects, which would then induce further investment and national income would continue to grow like a Topsy. Such a belief is a great illusion, however.

In actual practice, the interaction of multiplier and acceleration does not work for growth; at best, it is responsible for fluctuations on the path of movement of national income to higher and higher levels; that is, it does not work for growth; at best, it is responsible for fluctuations on the path of movement of national income from one level to another. The interaction, no doubt, shows significant cyclical effects but it has overlooked the other factors which work in the actual determination of the total income effect of the multiplier and acceleration principles.

Multiplier model

The concept of multiplier is derived from the concept of MPC.

It refers to effect of change in Autonomous spending on aggregate income through induced consumption expenditure.

It is the amount by which equilibrium output level changes when autonomous spending (A) changes by 1 unit. It is the ratio of change in equilibrium output level to a change in Autonomous Spending (A).

$\alpha = \Delta Y / \Delta A$

Government Expenditure Multiplier:

It is the rate of change in equilibrium level of income as a result of change in Government expenditure.

(*a*) $\boxed{\alpha_G = \frac{1}{1-c}}$ → when T = 0

(*b*) $\boxed{\alpha_G = \frac{1}{1-c}}$ → when T = T*a*

(*c*) $\boxed{\alpha_G = \frac{1}{1-c+ct}}$ → when T = ty

***Proof*:**

(*a*) When T = 0

$$Y_1 = \frac{1}{1-c}(\bar{C} + \bar{I} + G_1)$$

$$Y_2 = \frac{1}{1-c}(\bar{C} + \bar{I} + G_2)$$

$$\boxed{\Delta Y = \frac{1}{1-c} \cdot \Delta G} \quad \text{or} \quad \boxed{\frac{\Delta Y}{\Delta G} = \frac{1}{1-c} = \alpha_G}$$

Proof:

(*b*) When T = T*a*

$$Y = \bar{C} + cY - c\bar{T} + \bar{I} + G$$

$$Y - cY = \bar{C} + \bar{I} + G - c\bar{T}$$

$$Y_1 = \frac{1}{1-c}[\bar{C} + \bar{I} + G_1 - c\bar{T}]$$

$$Y_2 = \frac{1}{1-c}[\bar{C} + \bar{I} + G_2 - c\bar{T}]$$

$$\boxed{\frac{\Delta Y}{\Delta G} = \frac{1}{1-c} = \alpha_G}$$

(*c*) When T = *t*Y

$$Y = \bar{C} + c(Y - tY) + \bar{I} + G$$

$$Y - cY + ctY = \bar{C} + \bar{I} + G$$

$$Y_1 = \frac{1}{1-c+ct}[\bar{C} + \bar{I} + G_1] \qquad ...(i)$$

$$Y_2 = \frac{1}{1-c+ct}[\bar{C} + \bar{I} + G_2] \qquad ...(ii)$$

$$\boxed{\frac{\Delta Y}{\Delta G} = \frac{1}{1-c+ct} = \alpha_G > 0}$$

Increase in autonomous spending (ΔA) causes multiple increase in the equilibrium output and income level and the value of its multiple is given by multiplier. However, the value of multiplier depends on MPC. Greater the value of MPC, greater is the value of multiplier because a larger fraction of additional income will be consumed. This will lead to an increase in demand.

The multiplier theory recognizes the fact that change in income due to change in investment is not instantaneous. It is a gradual process by which income changes. The process of change in income involves a time lag. Thus, the multiplier is a stage by stage computation of change in income resulting from a change in investment till the full effect of multiplier is not realized.

In Period 1

Assume Autonomous spending increases by $(\Delta\bar{A})$.

With Aggregate output remaining constant

$$AD > AO$$

Result → It will lead to decrease in inventories

In Period 2

Production will expand by $\Delta\bar{A}$

This increase in production will lead to an equal increase in income, and this increase in income in turn will lead to an increase in expenditure by $c.\Delta\bar{A}$.

Due to an increase in consumption expenditure, Aggregate demand will increase but as AO is constant.

$$\therefore \quad AD > AO$$

As a result, production in *3rd period* will increase by $c.\Delta\bar{A}$

In Period 3 → Production will increase by an amount equal to $c.\Delta\bar{A}$

Due to increase in production income will increase

As a result → In period 3 AD will increase by $c^2.\Delta\bar{A}$

Again AD > AO

Production in *period 4* will increase by $c^2.\Delta\bar{A}$

As MPC < 1, therefore increase in induced expenditure in every round will become smaller and smaller.

$$\therefore \quad c^2 < c$$

Thus, Induced expenditure in the third period will be less than the induced expenditure in the second period.

This can be explained numerically: (Numerical is for understanding)

Assume:

C = 50 + 0.5Y : Initially I = 0

∴ Equilibrium income level is 100

Y = C + I

Y = 50 + 0.5Y + 0

0.5Y = 50

Y = 100

Assume in period 1 Autonomous spending e.g. Investment increases by 100

with MPC = 0.5 : AD increase to 200

with AO = 100	Y = 50 + 0.5Y + 100
AD = 200	= 50 + 0.5(100)+ 100
AD > AO	= 200

∴ *in period 3 :* AO will increase by 200

As a result income level will increase and thus expenditure will increase by $c.\Delta I = 0.5(100) = 50$

∴ AD = 200 + 50 = 250

with AO = 200

AD > AO

Therefore, income will increase from 200 to 250.

Due to increase in income AD with an increase by $c^2 . \Delta I = 0.5 \times 0.5 \times 100 = 25$

∴ AD = 250 + 25 = 275

Thus, AD (275) > AO (250) by 25

∴ $c^2 < c \quad (0.25 < 0.5)$

∴ Expenditure in period 3 will increase by a lesser amount than the increase in expenditure in period 2(50)

[Period 2 : 250 – 200 = 50]

[Period 3 : 275 – 250 = 25]

The gap between expenditure and income level will go on decreasing till the income level Y = 300 is reached. This is because,

$$\alpha = \frac{1}{1-c} = \frac{\Delta Y}{\Delta I}$$

$$\Rightarrow \quad \frac{1}{1-0.5} = \frac{\Delta Y}{100}$$

$$\Rightarrow \quad \Delta Y = 200$$

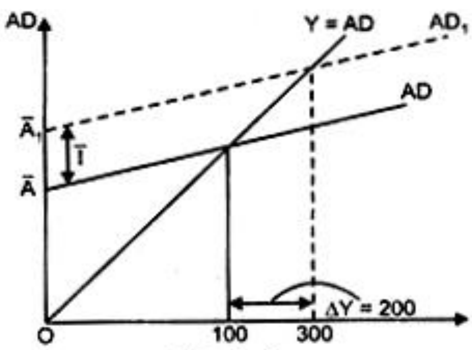


FIG. 7. 1

Initial income level = 100

ΔY = 200

∴ Total increase in income = 300

Period	Increase in Demand	Increase in Production	Total increase in income
1	$\Delta\bar{A}$	$\Delta\bar{A}$	$\Delta\bar{A}$
2	$c\Delta\bar{A}$	$c\Delta\bar{A}$	$(1+c)\Delta\bar{A}$
3	$c^2\Delta\bar{A}$	$c^2\Delta\bar{A}$	$(1+c+c^2)\Delta\bar{A}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$c^{n-1}\Delta\bar{A}$	$c^{n-1}\Delta\bar{A}$	$(1+c+c^2+...+c^{n-1})\Delta\bar{A}$

$$\Delta AD = \Delta\bar{A} + c\Delta\bar{A} + c^2\Delta\bar{A} + ... + c^{n-1}\Delta\bar{A} \quad ...(i)$$

$$\Delta Y \text{ or } \Delta AD = \Delta\bar{A}\,(1 + c + c^2 + ... + c^{n-1}) \quad ...(ii)$$

Multiply both sides by 'c' we get:

$$\Delta Y.c = \Delta\bar{A}\,(c + c^2 + ... + c^{n-1} + c^n) \quad ...(iii)$$

Subtract (*iii*) from (*i*) we get:

$$\Delta Y\,(1 - c) = \Delta\bar{A}(1 - c^n) \quad ...(iv)$$

or

$$\frac{\Delta Y}{\Delta\bar{A}} = \frac{1 - c^n}{1 - c} \quad \text{where } n \to \infty \quad ...(v)$$

As increase in income is given by geometric series, and

As $c < 1$, that is, the value of MPC lies between 0 and 1, therefore, the successive terms in the series become progressively smaller

$\therefore$ Equation (v) can be written as

$$\boxed{\Delta Y = \frac{1}{1-c}.\Delta\bar{A} = \alpha}$$

Thus, cumulative change in Aggregate spending equals Multiple increase in Autonomous spending.

$\boxed{\alpha = \frac{1}{1-c}}$ This equation shows that α (multiplier) is directly related to MPC.

Greater the value of MPC, greater is the value of multiplier (α)

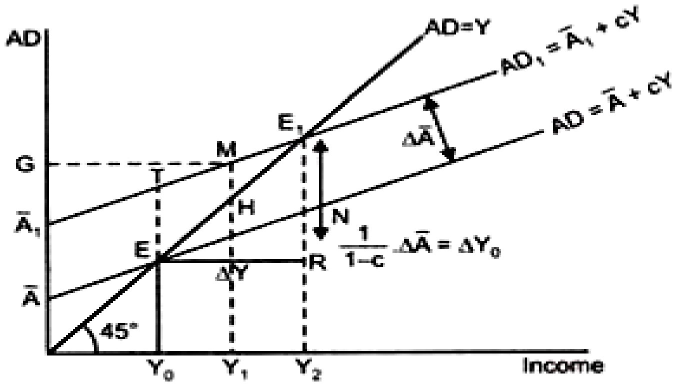
e.g. (i) MPC = 0.8

α = 1/1-0.8 = 5

(ii) If MPC = 0.6

α = 1/1-0.6 = 2.5

This is because high MPC implies that a greater fraction of additional increase in income will be consumed. Therefore, in every successive rounds (period), there will be greater increase in induced spending.



Initial equilibrium is at point E

equilibrium income level → OY_0 (Fig. 7.2)

If Autonomous spending $(\bar{A})$ increases from $\bar{A}$ to $\bar{A}_1$, AD curve shifts parallel upwards from AD to AD_1

Shift in AD means that at each income level AD will be increase by an amount $\Delta\bar{A}$

Where, $\boxed{\Delta\bar{A} \equiv \bar{A}_1 - \bar{A}}$

At initial output Y_0

Aggregate Demand > Aggregate output

$TY_0 > EY_0$

Result → Inventories will decrease

Firms will therefore expand production.

Assume production increases to Y_1

This will lead to rise in induced expenditure

Result – Aggregate demand increases to AD_1

But at this output AD > AO by MH, that is, there will still exist a gap between AD and AO by MH.

To build up this gap again production will increase. As a result, the gap between aggregate demand and output will decrease (MH < TE) because MPC < 1

This process will continue till a balance between AD and Aggregate output is not restored. This is at point E1

At $E_1 \rightarrow$ Aggregate Demand = Aggregate Output

Equilibrium level of income will increase from OY_1 to OY_2 $(OY_2 > OY_0)$

Total change in income level $\boxed{\Delta Y_0 = Y_2 - Y_0}$

Thus, increase in income is a result of increase in autonomous spending $\Delta\bar{A}$. However, the magnitude of increase in income will depend on:

1. $\Delta\bar{A} \rightarrow$ Greater the increase in $\Delta\bar{A}$, greater will be the increase in income level.
2. MPC $\rightarrow$ Greater the MPC, *i.e.*, steeper the AD curve, higher will be the increase in income.

Cobweb model

Cobweb theory is the idea that price fluctuations can lead to fluctuations in supply which cause a cycle of rising and falling prices.

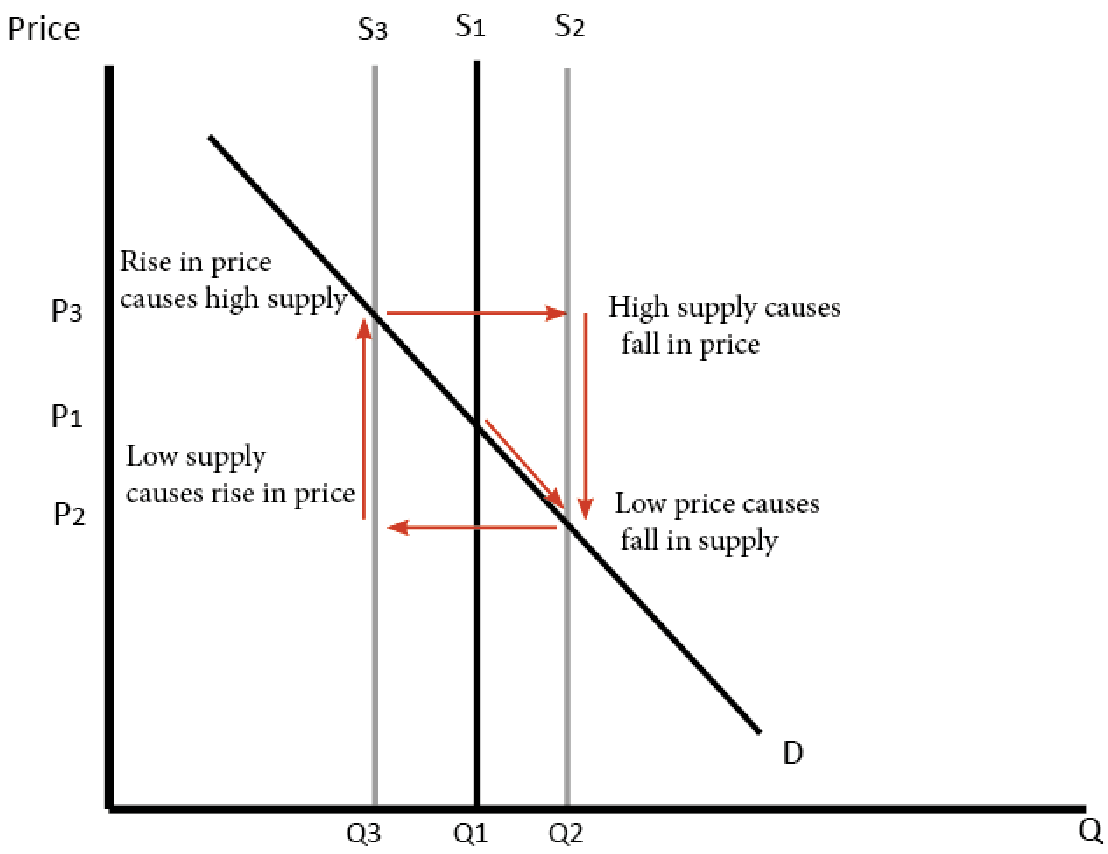
In a simple cobweb model, we assume there is an agricultural market where supply can vary due to variable factors, such as the weather.

The cobweb model purports to explain persistent fluctuations of prices in selected agricultural markets. It was first developed in the 1930s under static price expectations where the predicted price equalled actual price in the last period.

Cobweb models explain irregular fluctuations in prices and quantities that may appear in some markets. The key issue in these models is time, since the way in which expectations of prices adapt determines the fluctuations in prices and quantities. Cobweb models have been analysed by economists such as Ronald H. Coase, Wassily Leontief or Nicholas Kaldor. It was in Kaldor's paper on the subject, "A Classificatory Note on the Determinateness of Equilibrium", 1934, where the analysis of these models became of great interest, and where the phenomenon took the name of Cobweb theorem. Four years later, in 1938, economist Mordecai Ezekiel wrote the paper “The Cobweb Theorem”, which gave the phenomenon and its particular diagrams popularity.

Assumptions of Cobweb theory

1. In an agricultural market, farmers have to decide how much to produce a year in advance – before they know what the market price will be. (supply is price inelastic in short-term)
2. A key determinant of supply will be the price from the previous year.
3. A low price will mean some farmers go out of business. Also, a low price will discourage farmers from growing that crop in the next year.
4. Demand for agricultural goods is usually price inelastic (a fall in price only causes a smaller % increase in demand)



If there is a very good harvest, then supply will be greater than expected and this will cause a fall in price.

However, this fall in price may cause some farmers to go out of business. Next year farmers may be put off by the low price and produce something else. The consequence is that if we have one year of low prices, next year farmers reduce the supply.

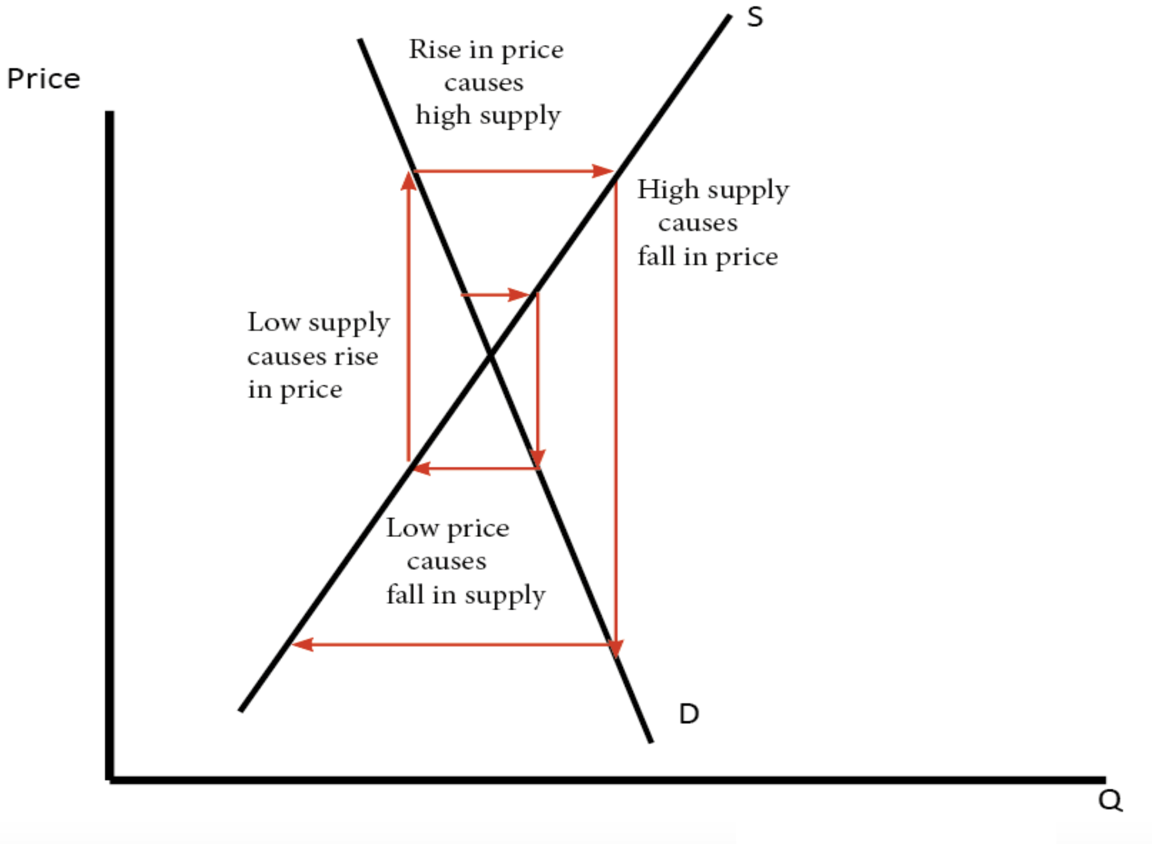
If supply is reduced, then this will cause the price to rise.

If farmers see high prices (and high profits), then next year they are inclined to increase supply because that product is more profitable.

In theory, the market could fluctuate between high price and low price as suppliers respond to past prices.

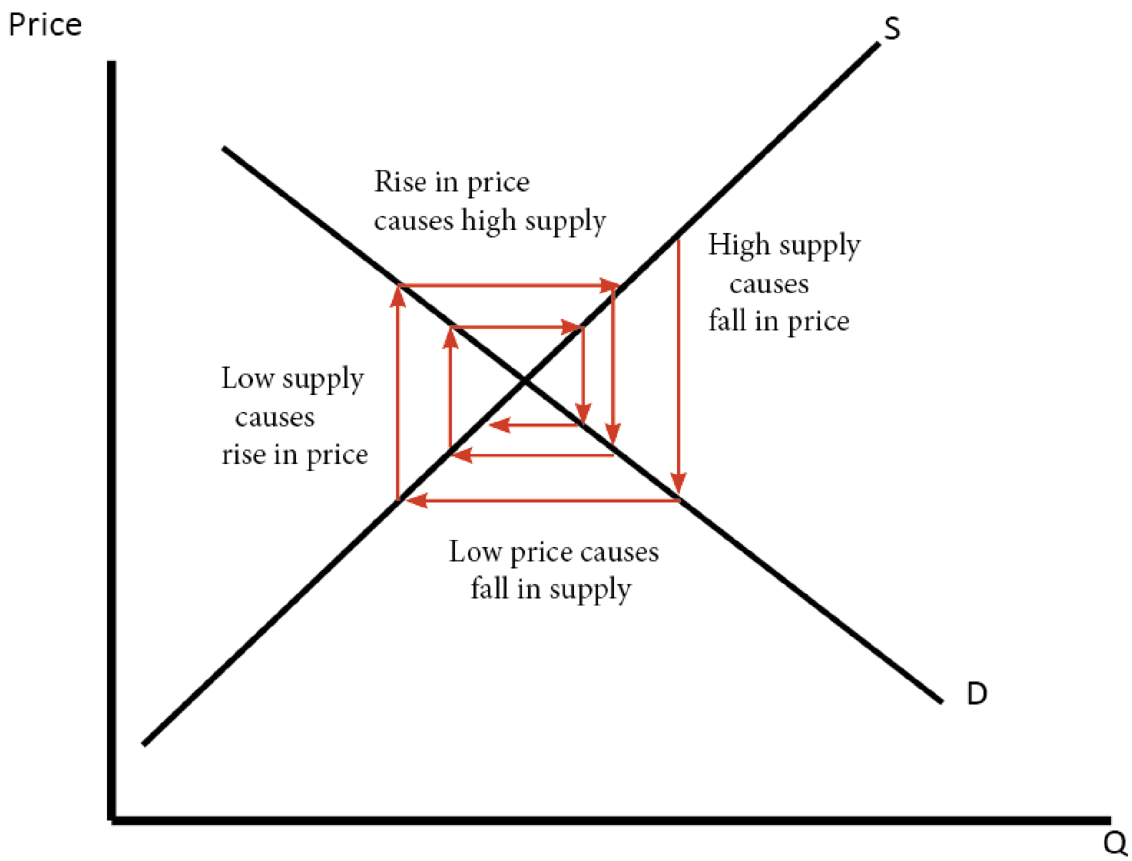
Cobweb theory and price divergence

Price will diverge from the equilibrium when the supply curve is more elastic than the demand curve, (at the equilibrium point)



If the slope of the supply curve is less than the demand curve, then the price changes could become magnified and the market more unstable.

Cobweb theory and price convergence



At the equilibrium point, if the demand curve is more elastic than the supply curve, we get the price volatility falling, and the price will converge on the equilibrium.

Limitations of Cobweb theory

Rational expectations. The model assumes farmers base next years supply purely on the previous price and assume that next year's price will be the same as last year (adaptive expectations). However, that rarely applies in the real world. Farmers are more likely to see it as a 'good' year or 'bad year and learn from price volatility.

Price divergence is unrealistic and not empirically seen. The idea that farmers only base supply on last year's price means, in theory, prices could increasingly diverge, but farmers would learn from this and pre-empt changes in price.

It may not be easy or desirable to switch supply. A potato grower may concentrate on potatoes because that is his speciality. It is not easy to give up potatoes and take to aubergines.

Other factors affecting price. There are many other factors affecting price than a farmers decision to supply. In global markets, supply fluctuations will be minimized by the role of importing from abroad. Also, demand may vary. Also, supply can vary due to weather factors.

Buffer stock schemes. Governments or producers could band together to limit price volatility by buying surplus.

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